VIRO THEOREM AND TOPOLOGY OF REAL AND COMPLEX COMBINATORIAL HYPERSURFACES*

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Abstract

We introduce a class of combinatorial hypersurfaces in the complex projective space. They are submanifolds of codimension 2 in $\mathbb{C}P^n$ and are topologically "glued" out of algebraic hypersurfaces in $(\mathbb{C}^*)^n$. Our construction can be viewed as a version of the Viro gluing theorem, relating topology of algebraic hypersurfaces to the combinatorics of subdivisions of convex lattice polytopes. If a subdivision is convex, then according to the Viro theorem a combinatorial hypersurface is isotopic to an algebraic one. We study combinatorial hypersurfaces resulting from non-convex subdivisions of convex polytopes, show that they are almost complex varieties, and in the real case, they satisfy the same topological restrictions (congruences, inequalities etc.) as real algebraic hypersurfaces.

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Introduction

Topology of real algebraic varieties, brought to the wide mathematical audience by D. Hilbert in his 16th problem, has been studied from two sides, looking for restrictions to geometric and topological properties of real algebraic varieties, and constructing varieties with prescribed properties (a lot of material, but, certainly not all, can be found in the surveys [7, 14, 33, 38]). Analyzing a gap between restrictions and constructions, O. Viro [33] suggested a concept of "flexible" curves, smooth surfaces in $\mathbb{C}P^2$ having certain topological properties of real algebraic curves, and asked for their classification in comparison with the classification of real algebraic curves. We mention a recent active study of real pseudo-holomorphic curves [10, 22, 23, 37] as a further development of this program.

In the present paper we suggest another source for flexible curves, or, more generally, flexible varieties which we call *combinatorial hypersurfaces*. These varieties are codimension 2 submanifolds in the complex projective space, they are conjugation invariant and their real parts are codimension 1 submanifolds in the real projective space. Our approach is based on the Viro construction of real algebraic varieties with prescribed topology [30, 31, 34, 36] (see also [13], 11.5, [19, 25]), which relates topology of real algebraic varieties to the combinatorics of Newton polytopes and their *convex* lattice subdivisions. The Viro gluing theorem states that the result of the gluing procedure is isotopic to a real algebraic variety under the condition of convexity of the subdivision used in the construction. We remove the convexity condition and show that the Viro construction modified in this way still can be performed and produces combinatorial hypersurfaces. Then we show that combinatorial hypersurfaces obey almost all known topological restrictions to real algebraic varieties. These results can be viewed as the first steps in the study of the following questions: how far are combinatorial hypersurfaces from the algebraic ones, and how crucial is the convexity condition in the Viro theorem ?

We preface the main material with illustrating examples and more detailed statement of the problem and our results.

Consider an example of the Viro construction. Let $T_d \subset \mathbb{R}^2$, $d \in \mathbb{N}$, be the triangle with vertices $(0,0)$, $(0,d)$, $(d,0)$,

$$\tau: \quad T_d = \Delta_1 \cup ... \cup \Delta_N$$

a triangulation with the set of vertices $V \subset \mathbb{Z}^2$, and $\sigma : V \to \{\pm 1\}$ any function. Out of this combinatorial data we construct a piecewise-linear plane curve. Denote by $T_d^{(1)}$, $T_d^{(2)}$ and $T_d^{(3)}$ the copies of T_d under reflections with respect to the coordinate axes and the origin. Take in $T_d^{(1)}$, $T_d^{(2)}$ and $T_d^{(3)}$ triangulations symmetric to τ, and define σ at the vertices of new triangulations by

$$\sigma(\varepsilon_1 i, \varepsilon_2 j) = \varepsilon_1^i \varepsilon_2^j \sigma(i,j), \quad (i,j) \in V, \quad \varepsilon_1, \varepsilon_2 = \pm 1.$$

Now in any triangle of the triangulation of $T_d \cup T_d^{(1)} \cup T_d^{(2)} \cup T_d^{(3)}$, having vertices with different values of σ, we draw the midline separating the vertices with different

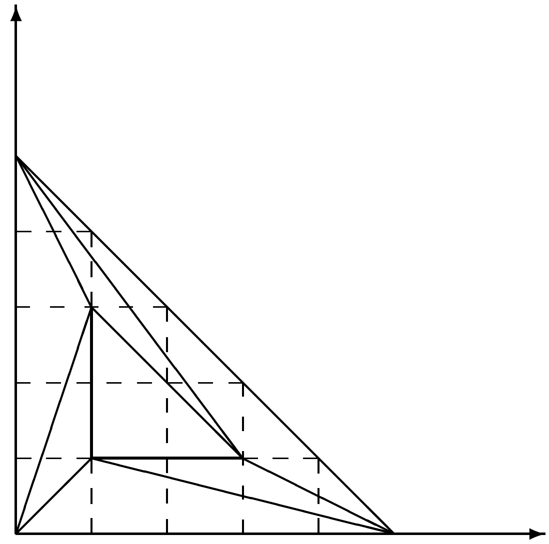

Figure 1: A non-convex triangulation

signs. The union $C(\tau,\sigma)$ of all these midlines is a broken line homeomorphic to a disjoint union of circles and segments. Introduce the following maps:

$$\Phi : T_d \cup T_d^{(1)} \cup T_d^{(2)} \cup T_d^{(3)} \to \mathbb{R}P^2, \quad \Psi : \mathrm{Int}(T_d \cup T_d^{(1)} \cup T_d^{(2)} \cup T_d^{(3)}) \to \mathbb{R}^2,$$

where Φ is continuous onto, identifying antipodal points on $\partial(T_d \cup T_d^{(1)} \cup T_d^{(2)} \cup T_d^{(3)})$, and Ψ is a homeomorphism. We call the curves $\Phi(C(\tau,\sigma)) \subset \mathbb{R}P^2$ and $\Psi(C(\tau,\sigma) \cap \mathrm{Int}(T_d \cup T_d^{(1)} \cup T_d^{(2)} \cup T_d^{(3)})) \subset \mathbb{R}^2$ *projective* and *affine T-curves of degree d*, respectively.

The Viro theorem states that a projective (resp., affine) T-curve of degree d is isotopic in $\mathbb{R}P^2$ (resp., in $\mathbb{R}^2$) to a nonsingular algebraic projective (resp., affine) curve of degree d, providing that the triangulation τ is *convex*, *i.e.*, there exists a convex piecewise-linear function $\nu : T_d \to \mathbb{R}$, whose linearity domains are $\Delta_1, \dots, \Delta_N$ (sometimes such triangulations are called *regular* or *coherent*; see [39] and [13]). The Viro theorem, in fact, endows the combinatorial broken line $C(\tau,\sigma)$ with a rich structure, which yields a number of restrictions on the topology of $C(\tau,\sigma)$ (see an account of known results in [27, 32, 33, 38]).

On the other hand, there exist *non-convex* triangulations. The simplest example is shown in Figure 1 (see, for instance, [5]). Moreover, no efficient criterion for the convexity of a triangulation is known. There are examples of T-curves beyond the range of known algebraic curves [28], and there is some similarity between T-curves and algebraic curves: up to degree 6 any projective T-curve is isotopic to an algebraic one of the same degree [8], and *vice versa*, any nonsingular algebraic curve of degree at most 6 in $\mathbb{R}P^2$ is isotopic to a projective T-curve of the same degree. T-curves satisfy some consequences of the Bézout theorem [8], the Harnack inequality [18, 15], and the complex orientation formula [24]. Maximal T-curves satisfy the Ragsdale type inequality [15], which is not proved for maximal real algebraic curves. It is

natural to ask whether any T-curve is isotopic to an algebraic curve of the same degree, and if not, how far T-curves may differ from algebraic ones.

In the general Viro construction, which applies to any dimension, one starts with a convex lattice subdivision $P = \Delta_1 \cup \ldots \cup \Delta_N$ of an n-dimensional convex lattice polytope P in $(\mathbb{R}_+)^n$ (where $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$) and an appropriate collection of polynomials $F_1, \ldots, F_N$. Then the Viro construction produces an algebraic hypersurface A in the toric variety $\mathrm{Tor}_{\mathbb{C}}(P)$ associated with P. The Newton polytope of A is P, and the topology of the complex part of A (and of the real part of A if the polynomials $F_1, \ldots, F_N$ are real) is described in terms of topology of zero point sets of $F_1, \ldots, F_N$. Namely, the complex (resp., real) part of A is, in a sense, "glued" out of the complex (resp., real) zero sets of $F_1, \ldots, F_N$.

The zero sets of the initial polynomials in the Viro construction can be "glued" even if the subdivision is not convex (as it was done above in the case of T-curves). However, in this situation the Viro theorem does not guarantee that the result carries an algebraic structure. In the present paper we study the following question: what can be said about the result of the "gluing" in the case of *non-convex* subdivisions of Newton polytopes ?

Let us restrict ourselves to the situation when P is the simplex T_d^n in $\mathbb{R}^n$ with vertices $(0, 0, ..., 0)$, $(d, 0, ..., 0)$, ..., $(0, ..., 0, d)$. Note that the toric variety $\mathrm{Tor}_{\mathbb{C}}(T_d^n)$ associated with T_d^n is $\mathbb{C}P^n$. We show that the result of "gluing" of the complex zero sets of the polynomials $F_1, \ldots, F_N$ is a codimension 2 piecewise-smooth submanifold in $\mathbb{C}P^n$. We call this piecewise-smooth submanifold a *(complex) combinatorial hypersurface* (briefly, *C-hypersurface*) *of degree d in* $\mathbb{C}P^n$. If the initial polynomials are real, then the resulting C-hypersurface is invariant under the complex conjugation in $\mathbb{C}P^n$. In this case, the C-hypersurface is called *real.* The real point set of a real C-hypersurface is a piecewise-analytic hypersurface in $\mathbb{R}P^n$ and is the result of "gluing" of the real zero sets of the polynomials $F_1, \ldots, F_N$.

It is natural to compare the class of C-hypersurfaces of degree d in $\mathbb{C}P^n$ with the class of algebraic hypersurfaces of the same degree. The following questions arise.

1. *Is any C-hypersurface of degree d in $\mathbb{C}P^n$ homeomorphic (or isotopic) to an algebraic hypersurface of the same degree ?*

2. *Is the real part of any real C-hypersurface of degree d in $\mathbb{C}P^n$ homeomorphic (or isotopic in $\mathbb{R}P^n$) to the real part of a real algebraic hypersurface of the same degree ?*

3. *Is any real C-hypersurface of degree d in $\mathbb{C}P^n$ equivariantly isotopic to a real algebraic hypersurface of the same degree ?*

In the present work we try to do the first steps to answering these questions. We show that any C-hypersurface of degree d in $\mathbb{C}P^n$ can be smoothed and carries an almost complex structure. We prove that C-hypersurfaces of degree d in $\mathbb{C}P^n$ share a lot of topological properties with algebraic hypersurfaces of degree d in $\mathbb{C}P^n$. In particular, the answer to the first question is positive in the case $n = 2$. Moreover,

any real C-curve in $\mathbb{C}P^2$ can be smoothed out into a *flexible* curve in the sense of [33]. As a corollary, we obtain that arbitrary T-curves satisfy all *topological* restrictions known for real algebraic curves. We also prove that any C-surface of degree d in $\mathbb{C}P^3$ is homeomorphic to an algebraic surface of the same degree.

The material of section 1, where we describe the construction of C-hypersurfaces, is basically contained in [30]. We provide here all details, since our setting is a little bit different. In section 2 we study the topological properties of (complex) C-hypersurfaces of degree d in $\mathbb{C}P^n$ and of double coverings of $\mathbb{C}P^n$ ramified along C-hypersurfaces. In particular, we reprove the projective complex version of Viro's theorem (see [30]) which claims that a C-hypersurface (resp., real C-hypersurface) of degree d in $\mathbb{C}P^n$ is isotopic (resp., equivariantly isotopic) to an algebraic hypersurface of the same degree, provided the subdivision is convex. In the case of arbitrary (not necessarily convex) subdivisions, we prove the following statements.

- A C-hypersurface of degree d in $\mathbb{C}P^n$ is an orientable manifold, homologous to an algebraic hypersurface of degree d in $\mathbb{C}P^n$.
- Any C-hypersurface M in $\mathbb{C}P^n$ is isotopic to a close smooth manifold M_{sm} of codimension 2 in $\mathbb{C}P^n$. If M is real, the isotopy can be made equivariant.
- Given a (real) C-hypersurface M, the tangent bundle to its smoothing M_{sm} is (equivariantly) isotopic to a (equivariant) bundle of complex hyperplanes. In particular, M_{sm} possesses an (equivariant) almost complex structure.
- A C-hypersurface in $\mathbb{C}P^n$ is simply connected if $n > 2$.
- For a C-hypersurface M of degree d in $\mathbb{C}P^n$, one has $\pi_1(\mathbb{C}P^n \backslash M) = \mathbb{Z}/d\mathbb{Z}$ if $n \geq 2$.
- Let $M \subset \mathbb{C}P^n$ be a C-hypersurface and $M' \subset \mathbb{C}P^n$ a nonsingular algebraic hypersurface, both of degree d. Then $H_*(M)$ and $H_*(M')$ are isomorphic as graded groups.
- Let n be a positive odd number, M a C-hypersurface of degree d in $\mathbb{C}P^n$ and M' a nonsingular algebraic hypersurface of degree d in $\mathbb{C}P^n$. Then the lattices $(H_{n-1}(M), B_M)$ and $(H_{n-1}(M'), B_{M'})$ (where $B_M : H_{n-1}(M) \times H_{n-1}(M) \to \mathbb{Z}$ and $B_{M'} : H_{n-1}(M') \times H_{n-1}(M') \to \mathbb{Z}$ are the intersection forms of M and M', respectively) are isomorphic.
- A C-surface of degree d in $\mathbb{C}P^3$ is homeomorphic to a nonsingular algebraic surface of degree d in $\mathbb{C}P^3$.
- Any C-curve in $\mathbb{C}P^2$ is isotopic to an algebraic curve of the same degree.

Section 3 is devoted to topology of real C-hypersurfaces in $\mathbb{C}P^n$. We prove for them the generalized Harnack inequality and the Gudkov-Rokhlin and Gudkov-Krahnov-Kharlamov congruences. For real C-surfaces in $\mathbb{C}P^3$, we also prove inequalities similar to the Comessatti ones.

Remark 0.1 *An interesting question concerns the existence of a Hodge-like decomposition in the cohomology of C-hypersurfaces. A closely related question is whether the analog of Petrovsky-Oleinik inequalities is true for real C-hypersurfaces in* $\mathbb{C}P^n$ *?*

1 Construction of C-hypersurfaces

1.1 Notations and definitions

Further on the term *polytope* (*polygon*) means a convex lattice polytope (polygon) in the nonnegative orthant $\mathbb{R}^n_+$ of $\mathbb{R}^n$, $n \ge 2$.

Given a polynomial

$$F = \sum_{i_1,...,i_n} A_{i_1...i_n} z_1^{i_1} \cdot ... \cdot z_n^{i_n} \ ,$$

by $\Delta(F)$ we denote its Newton polytope, *i.e.*, the convex hall of the set

$$\{(i_1, ..., i_n) \in \mathbb{R}^n \ : \ A_{i_1...i_n} \neq 0\} \ .$$

The truncation of F on a face δ of $\Delta(F)$ is the polynomial

$$F^\delta = \sum_{(i_1,...,i_n)\in\delta} A_{i_1...i_n} z_1^{i_1} \cdot ... \cdot z_n^{i_n} \ .$$

A polynomial $F \in \mathbb{C}[z_1, ..., z_n]$ is called *non-degenerate*, if F and any truncation F^δ on a proper face δ of $\Delta(F)$ has a nonsingular zero set in $(\mathbb{C}^*)^n$ (cf. [30]).

1.2 Extension of the moment map

Let $\Delta \subset \mathbb{R}^n$ be a polytope, and let $\mu_\Delta : (\mathbb{R}^*_+)^n \to I(\Delta)$ be the moment map (see [1, 2, 12], [13], 6.1), where $\mathbb{R}^*_+ = \{x \in \mathbb{R}, x > 0\}$ and $I(\Delta)$ is the complement in Δ of the union of all its proper faces,

$$\mu_\Delta(x_1, ..., x_n) = \frac{\sum_{(i_1,...,i_n)\in\Delta} x_1^{i_1} ... x_n^{i_n} \cdot (i_1, ..., i_n)}{\sum_{(i_1,...,i_n)\in\Delta} x_1^{i_1} ... x_n^{i_n}} \ . \qquad (1)$$

Split the complex torus in the product $(\mathbb{C}^*)^n = (\mathbb{R}^*_+)^n \times (S^1)^n$:

$$(z_1, ..., z_n) \in (\mathbb{C}^*)^n \ \mapsto \ (|z_1|, ..., |z_n|) \in (\mathbb{R}^*_+)^n, \ \left(\frac{z_1}{|z_1|}, ..., \frac{z_n}{|z_n|}\right) \in (S^1)^n \ .$$

Note that the inverse map $(\mathbb{R}^*_+)^n \times (S^1)^n \to (\mathbb{C}^*)^n$ naturally extends to a surjection $\theta : \mathbb{R}^n_+ \times (S^1)^n \to \mathbb{C}^n$. Put

$$\mathbb{C}I(\Delta) = \theta(I(\Delta) \times (S^1)^n) \subset \mathbb{C}^n, \quad \mathbb{C}\Delta = \theta(\Delta \times (S^1)^n) \subset \mathbb{C}^n \ .$$

Proposition 1.1 *The complexification $\mathbb{C}\Delta$ of Δ is a singular PL-manifold with boundary. The singular set of $\mathbb{C}\Delta$ is the union of $\mathbb{C}\delta$ over all faces $\delta \subset \Delta$, which are intersections of Δ with coordinate planes of dimension $> n - \dim\Delta + \dim\delta$. The real part $\mathbb{R}\Delta$ of $\mathbb{C}\Delta$ is the union of Δ with all its symmetric copies with respect to the coordinate hyperplanes.*

Proof. Straightforward. $\square$

Define *the extended moment map* $\mathbb{C}\mu_\Delta : (\mathbb{C}^*)^n \to \mathbb{C}I(\Delta)$ by

$$\mathbb{C}\mu_\Delta(x_1v_1, ..., x_nv_n) = \theta(\mu_\Delta(x_1, ..., x_n)\ , (v_1, ..., v_n)), \tag{2}$$
$$(x_1, ..., x_n) \in (\mathbb{R}^*_+)^n,\ (v_1, ..., v_n) \in (S^1)^n\ ,$$
$$\theta(\mu_\Delta(x_1, ..., x_n), (v_1, ..., v_n)) \in \theta(I(\Delta) \times (S^1)^n) = \mathbb{C}I(\Delta)\ .$$

As an easy consequence of classical results we obtain the following statement.

Proposition 1.2 *The map $\mathbb{C}\mu_\Delta$ is surjective and commutes with the complex conjugation* Conj. *It is a diffeomorphism when* $\dim\Delta = n$. *The real part of $\mathbb{C}I(\Delta)$ is the image of $(\mathbb{R}^*)^n$.*

Below we use the following interaction of the extended moment map with the action of $\mathbb{Z}^n$ in $\mathbb{R}^n$

$$\overline{a} = (a_1, ..., a_n) \in \mathbb{Z}^n,\ \overline{t} = (t_1, ..., t_n) \in \mathbb{R}^n \quad \mapsto \quad \overline{a} + \overline{t} = (a_1 + t_1, ..., a_n + t_n) \in \mathbb{R}^n\ ,$$

and the actions of $SL(n, \mathbb{Z})$ in $\mathbb{R}^n$ and $(\mathbb{R}^*)^n$

$$A \in SL(n, \mathbb{Z}),\ \overline{x} \in \mathbb{R}^n \quad \mapsto \quad A\overline{x} \in \mathbb{R}^n\ ,$$

$$A \in SL(n, \mathbb{Z}),\ \overline{x} = (x_1, ..., x_n) \in (\mathbb{R}^*)^n\ \mapsto\ \overline{x}^A = \left(\prod_{i=1}^n x_i^{a_{i1}}, ..., \prod_{i=1}^n x_i^{a_{in}}\right)\ .$$

Proposition 1.3 *Let $\Delta \subset \mathbb{R}^n$ be a polygon. If $\overline{a} = (a_1, ..., a_n) \in \mathbb{Z}^n$, $A \in SL(n, \mathbb{Z})$, and $\overline{a} + A\Delta$ lies in the nonnegative orthant $\mathbb{R}^n_+$, then*

$$\mathbb{C}\mu_{\overline{a}+A\Delta}(\overline{z}) = (\overline{a} + A\mu_\Delta(\overline{x}^A), \overline{v})\ , \tag{3}$$

where $\overline{z} = (\overline{x}, \overline{v}) \in (\mathbb{R}^*_+)^n \times (S^1)^n$.

Proof. The result follows from the definition

$$\mathbb{C}\mu_{\overline{a}+A\Delta}(\overline{z}) = (\mu_{\overline{a}+A\Delta}(\overline{x}), \overline{v})\ ,$$

and the classically known relation

$$\mu_{\overline{a}+A\Delta}(\overline{x}) = \overline{a} + A\mu_\Delta(\overline{x}^A) \quad \square$$

1.3 Real and complex chart of a polynomial

Let $F \in \mathbb{C}[z_1, ..., z_n]$ be a non-degenerate polynomial with Newton polytope $\Delta(F) = \Delta$. The closure $\mathbb{C}Ch(F) \subset \mathbb{C}\Delta$ of the set $\mathbb{C}\mu_\Delta(\{F = 0\} \cap (\mathbb{C}^*)^n)$ is called the *(complex) chart* of the polynomial F. If F is real then $\mathbb{R}Ch(F) = \mathbb{C}Ch(F) \cap \mathbb{R}\Delta$ is called the *real chart* of F.

This definition is a key ingredient of the Viro construction (see [30, 25], cf. [29]).

Proposition 1.4 *Suppose that $\Delta \subset (\mathbb{R}^*_+)^n$. Then the set $\mathbb{C}Ch(F)$ is a PL-submanifold in $\mathbb{C}\Delta$ of codimension 2 with boundary $\partial\mathbb{C}Ch(F) = \mathbb{C}Ch(F) \cap \partial\mathbb{C}\Delta$. It is smooth in $\mathbb{C}I(\Delta)$, and, for any proper face δ of Δ of positive dimension,*

$$\mathbb{C}Ch(F) \cap \mathbb{C}\delta = \mathbb{C}Ch(F^\delta) \ . \tag{4}$$

If F is real then $\mathbb{C}Ch(F)$ is invariant with respect to Conj, *and $\mathbb{R}Ch(F)$ is a PL-submanifold in $\mathbb{R}\Delta$ of codimension 1 with boundary $\partial\mathbb{R}Ch(F) = \mathbb{R}Ch(F) \cap \partial\mathbb{R}\Delta$.*

Remark 1.5 *If Δ intersects with coordinate hyperplanes, then the statement of Proposition 1.4 holds true when substituting $\mathbb{C}\Delta \backslash \mathrm{Sing}(\mathbb{C}\Delta)$ and $\mathbb{R}\Delta \backslash \mathrm{Sing}(\mathbb{C}\Delta)$ for $\mathbb{C}\Delta$ and $\mathbb{R}\Delta$, respectively, but we will not use this below.*

Proof of Proposition 1.4.

(i) If $\dim \Delta = n$ then by Proposition 1.2, $\mathbb{C}Ch(F) \cap \mathbb{C}I(\Delta)$ is a smooth manifold of codimension 2.

So, we have to consider only the behavior of $\mathbb{C}Ch(F)$ on $\partial\mathbb{C}\Delta$.

(ii) Assume that $\dim \Delta < n$ and $A \in SL(n, \mathbb{Z})$ is such that $A\Delta \subset (\mathbb{R}^*_+)^n$ is parallel to the coordinate hyperplane $i_n = 0$. By (3) there is a diffeomorphism which takes the pair $(\mathbb{C}\Delta, \mathbb{C}Ch(F(\overline{z})))$ onto the pair $(\mathbb{C}(A\Delta), \mathbb{C}Ch(F(\overline{z}^A)))$. Now, if $i_n\big|_\Delta = a$, then $\Delta' = \Delta - (0, ..., 0, a) \subset \{i_n = 0\}$, $F(\overline{z}) = z_n^a F'(\overline{z})$, and by formula (3) the pair $(\mathbb{C}\Delta, \mathbb{C}Ch(F))$ naturally splits into the product $(C\Delta', \mathbb{C}Ch(F')) \times S^1$, thereby one reduces the dimension of the ambient space.

(iii) To prove (4) (for $\dim \Delta = n$), we note that any point $\overline{z}^0 \in \mathbb{C}Ch(F) \cap \partial\mathbb{C}\Delta$ is a limit of $\mathbb{C}\mu_\Delta(\gamma(t))$ as $t > 0$, $t \to 0$, where

$$\gamma(t) = (\lambda_1 t^{k_1} + O(t^{k_1+1}), ..., \lambda_n t^{k_n} + O(t^{k_n+1}))$$

is a curve lying in $\{F = 0\} \cap (\mathbb{C}^*)^n$ with $\lambda_1 ... \lambda_n \neq 0$, $(k_1, ..., k_n) \in \mathbb{Z}^n \backslash \{0\}$. Let δ be the maximal (proper) face of Δ, where the function $(i_1, ..., i_n) \in \Delta \mapsto i_1 k_1 + ... + i_n k_n$ achieves its minimum (equal to l). Then

$$0 = F(\gamma(t)) = F^\delta(\lambda_1, ..., \lambda_n) t^l + O(t^{l+1}) \quad \Longrightarrow \quad F^\delta(\lambda_1, ..., \lambda_n) = 0 \ ,$$

$\gamma(t) = (\overline{x}, \overline{v}) \in (\mathbb{R}^*_+)^n \times (S^1)^n$, where

$$\overline{x} = (|\lambda_1| t^{k_1} + O(t^{k_1+1}), ..., |\lambda_n| t^{k_n} + O(t^{k_n+1})),\ \overline{v} = \left(\frac{\lambda_1}{|\lambda_1|} + O(t), ..., \frac{\lambda_n}{|\lambda_n|} + O(t)\right) .$$

So, one derives

$$\mathbb{C}\mu_\Delta(\gamma(t)) = (\mu_\Delta(\overline{x}),\overline{v}) = \left(\frac{\sum_{(i_1,...,i_n)\in\Delta}(i_1,...,i_n)x_1^{i_1}...x_n^{i_n}}{\sum_{(i_1,...,i_n)\in\Delta}x_1^{i_1}...x_n^{i_n}},\ \overline{v}\right)$$

$$= \left(\frac{t^l\cdot\sum_{(i_1,...,i_n)\in\delta}(i_1,...,i_n)|\lambda_1|^{i_1}...|\lambda_n|^{i_n}+O(t^{l+1})}{t^l\cdot\sum_{(i_1,...,i_n)\in\delta}|\lambda_1|^{i_1}...|\lambda_n|^{i_n}+O(t^{l+1})},\ \overline{v}\right) \quad \stackrel{t\to 0}{\longrightarrow} \quad \mathbb{C}\mu_\delta(\lambda_1,...,\lambda_n).$$

Hence $\mathbb{C}Ch(F)\cap\mathbb{C}\delta\subset\mathbb{C}Ch(F^\delta)$.

Now let $(\lambda_1,...,\lambda_n)\in(\mathbb{C}^*)^n$, $F^\delta(\lambda_1,...,\lambda_n)=0$, where δ is a proper face of Δ, and let $(k_1,...,k_n)\in k(\delta)$, where $k(\delta)\subset\mathbb{Z}^n\backslash\{0\}$ consists of vectors $(k_1,...,k_n)$ such that δ is the maximal face of Δ on which the function $(i_1,...,i_n)\in\Delta\mapsto i_1k_1+...+i_nk_n$ achieves its minimum. Then $F^\delta(\lambda_1t^{k_1},...,\lambda_nt^{k_n})=t^lF^\delta(\lambda_1,...,\lambda_n)=0$, $l=i_1k_1+...+i_nk_n$, $(i_1,...,i_n)\in\delta$, and there exist smooth functions $\varphi_1(t)=\lambda_1+O(t)$, ..., $\varphi_n=\lambda_n+O(t)$ such that the curve $\gamma(t)=(t^{k_1}\varphi_1(t),...,t^{k_n}\varphi_n(t))$ lies on $\{F=0\}$. Indeed,

$$F(\gamma(t)) = t^\lambda F^\delta(\varphi_1,...,\varphi_n)+t^{\lambda+1}G(t,\varphi_1,...,\varphi_2)=0$$
$$\iff \quad F^\delta(\varphi_1,...,\varphi_n)+tG(t,\varphi_1,...,\varphi_2)=0\ ,$$

hence the existence of $\varphi_1,...,\varphi_n$ defined for small $t>0$ follows from the implicit function theorem and the fact that $F^\delta=0$ is nonsingular in $(\mathbb{C}^*)^n$. The above argument shows that

$$\mathbb{C}\mu_\delta(\lambda_1,...,\lambda_n)=\lim_{t\to 0}\mathbb{C}\mu_\Delta(\gamma(t))\ .$$

(iv) Assume that $\dim\Delta=n$. Let $\delta\subset\Delta$ be a proper face, $\dim\delta=s>0$, and $\overline{w}=(y_1v_1^0,...,y_nv_n^0)\in\mathbb{C}I(\delta)\cap\mathbb{C}Ch(F)$, where $(y_1,...,y_n)\in I(\delta)$, $(v_1^0,...,v_n^0)\in(S^1)^n$. We will describe $\mathbb{C}Ch(F)$ in a neighborhood $U_{\overline{w}}$ of the point $\overline{w}$ in $\mathbb{C}\Delta$.

Represent $\mathbb{R}^n$ as $\mathbb{R}^s\times\mathbb{R}^{n-s}$ and denote by $\mathrm{pr}_s:\mathbb{R}^n\to\mathbb{R}^s$, $\mathrm{pr}_{n-s}:\mathbb{R}^n\to\mathbb{R}^{n-s}$ the natural projections. According to Proposition 1.3, we can assume that $\mathrm{pr}_{n-s}(\delta)$ is a point $(p_{s+1},...,p_n)$ with positive integers $p_{s+1},...,p_n$, so $\widetilde{\delta}=\mathrm{pr}_s(\delta)=\delta-(0,...,0,p_{s+1},...,p_n)$, and $\widetilde{\Delta}\backslash\widetilde{\delta}$ lies in the positive orthant. Then $F(z_1,...,z_n)=z_{s+1}^{p_{s+1}}...z_n^{p_n}\widetilde{F}(z_1,...,z_n)$, where

$$\widetilde{F}(z_1,...,z_n)=\widetilde{F}^{\widetilde{\delta}}(z_1,...,z_s)+\sum_{(i_1,...,i_n)\in\Lambda}A_{i_1...i_n}z_1^{i_1}...z_s^{i_s}\cdot M_{i_{s+1}...i_n}\ , \tag{5}$$

$$\Lambda=\{(i_1,...i_n)\in\widetilde{\Delta}\ :\ i_{s+1},...,i_n>0\}\cap\mathbb{Z}^n,\quad M_{i_{s+1}...i_n}=z_{s+1}^{i_{s+1}}...z_n^{i_n}\ .$$

Note that there are uniquely defined $x_1^0,...,x_s^0>0$ such that $(y_1,...,y_s)=\mu_{\widetilde{\delta}}(x_1^0,...,x_s^0)$. The condition on $\overline{z}=(\overline{x},\overline{v})\in(\mathbb{R}_+^*)^n\times(S^1)^n=(\mathbb{C}^*)^n$ such that $\mathbb{C}\mu_\Delta(\overline{x},\overline{v})\in U_{\overline{w}}$ can be expressed as follows. Clearly, for $\overline{v}=(v_1,...,v_n)$, it is

$$|v_1-v_1^0|<\varepsilon,\ ...,\ |v_n-v_n^0|<\varepsilon, \tag{6}$$

where $\varepsilon > 0$ is sufficiently small. For $\overline{x} = (x_1, ..., x_n)$, we have that $\mu_{\widetilde{\Delta}}(\overline{x})$, equal to

$$\frac{\sum_{(i_1,...,i_s)\in\widetilde{\delta}} x_1^{i_1}...x_s^{i_s} \cdot (i_1, ..., i_s, 0, ..., 0) + \sum_{(i_1,...,i_n\in\Lambda} x_1^{i_1}...x_s^{i_s} \cdot |M_{i_{s+1}...i_n}| \cdot (i_1, ..., i_n)}{\sum_{(i_1,...,i_s)\in\widetilde{\delta}} x_1^{i_1}...x_s^{i_s} + \sum_{(i_1,...,i_n\in\Lambda} x_1^{i_1}...x_s^{i_s} \cdot |M_{i_{s+1}...i_n}|} ,$$

is close to $(y_1, ..., y_s, 0, ..., 0)$. This means that in the latest expression the summands over Λ are small with respect to the sums over $\widetilde{\delta}$; hence $\mathrm{pr}_s(\mu_{\widetilde{\Delta}}(\overline{x}))$ is close to $\mu_{\widetilde{\delta}}(x_1, ..., x_s)$, which means that $(x_1, ..., x_s)$ is close to $(x_1^0, ..., x_s^0)$. Coming back to the sums over Λ, we obtain that $M_{i_{s+1}...i_n}$ must be close to zero, $(i_{s+1}, ..., i_n) \in \mathrm{pr}_{n-s}(\Lambda)$. So, without loss of generality the conditions on $\overline{x}$ can be expressed as

$$|x_l - x_l^0| < \varepsilon,\ l = 1, ..., s, \quad x_{s+1}^{i_{s+1}}...x_n^{i_n} < \varepsilon,\ (i_{s+1}, ..., i_n) \in \mathrm{pr}_{n-s}(\Lambda) . \tag{7}$$

Let $U' \subset (\mathbb{R}_+^*)^n$ and $U'' \subset (S^1)^n$ be given by (7) and (6), respectively, and note that $U' = \mathrm{pr}_s(U') \times \mathrm{pr}_{n-s}(U')$. Put $\sigma = \mathrm{pr}_{n-s}(\widetilde{\Delta})$. Then μ_σ takes $\mathrm{pr}_{n-s}(U')$ onto a set $U_0 \backslash \partial\sigma$, where U_0 is a neighborhood of the origin (a vertex of σ !) in σ.

Since $\widetilde{F}$ is non-degenerate, we can assume that

$$\frac{\partial \widetilde{F}}{\partial x_1}(x_1^0, ..., x_s^0, 0, ..., 0) = \frac{\partial \widetilde{F}^{\widetilde{\delta}}}{\partial x_1}(x_1^0, ..., x_s^0) \neq 0 .$$

Hence, by (5) and the implicit function theorem, the hypersurface $\{F = 0\}$ can be described in $U' \times U''$ (with, possibly, smaller ε) by equations

$$x_1 = x_1^0 + G, \quad v_1 = v_1^0 + H , \tag{8}$$

where G and H are vanishing at zero smooth functions of $x_2 - x_2^0$, ..., $x_s - x_s^0$, $v_2 - v_2^0$, ..., $v_n - v_n^0$, and $M_{i_{s+1}...i_n}$, $(i_{s+1}, ..., i_n) \in \mathrm{pr}_{n-s}(\Lambda)$, where $x_2, ..., x_n, v_2, ..., v_n$ are arbitrary satisfying (6) and (7).

Now we show that $U_{\overline{w}}$ is homeomorphic to $\mathrm{pr}_s(U') \times U_0 \times U''$ via an extension of $\mathbb{C}\mu_\Delta$. Indeed, $U_{\overline{w}} \backslash \partial\mathbb{C}\Delta$ is homeomorphic via $\mathbb{C}\mu_\Delta$ to $\mathrm{pr}_s(U') \times \mathrm{pr}_{n-s}(U') \times U'' \simeq \mathrm{pr}_s(U'') \times (U_0 \backslash \partial\sigma) \times U''$. As we saw, the points in $U_{\overline{w}} \cap \partial\Delta$ are $\lim_{t\to 0} \mathbb{C}\mu_\Delta(\overline{x}(t), \overline{v}(t))$ for all curves $\overline{x}(t) \in U'$, $\overline{v}(t) \in U''$, $t > 0$, such that $\lim_{t\to 0} \overline{v}(t) \in U''$, $\lim_{t\to 0}(x_1(t), ..., x_s(t)) \in \mathrm{pr}_s(U')$ and

$$x_{s+1}(t) = \lambda_1 t^{k_1} + O(t^{k_1+1}),\ ...,\ x_n(t) = \lambda_{n-s} t^{k_{n-s}} + O(t^{k_{n-s}+1}) .$$

Restriction (7) means, first, that the vector $(k_1, ..., k_{n-s})$ belongs to the cone generated by the interior normal vectors to the $(s-1)$-dimensional faces of σ, which contain the origin, and, second, that $\lambda_1^{i_{s+1}}...\lambda_{n-s}^{i_n} < \varepsilon$ for all $(i_{s+1}, ..., i_n) \in \mathrm{pr}_{n-s}(\Lambda)$ orthogonal to $(k_1, ..., k_{n-s})$. On the other hand, the same data determine a point $\lim_{t\to 0} \mu_\sigma(x_{s+1}(t), ..., x_n(t)) \in U_0 \cap \partial\sigma$. This gives a one-to-one correspondence between $U_{\overline{w}} \cap \partial\Delta$ and $\mathrm{pr}_s(U') \times (U_0 \cap \partial\sigma) \times U''$, which respects the combinatorial structure: the interior of a face $\psi \subset U_{\overline{w}} \cap \partial\Delta$ corresponds to $\mathrm{pr}_s(U') \times \mathrm{pr}_{n-s}(I\widetilde{\psi}) \times U''$, where $\widetilde{\psi} = \psi - (0, ..., 0, p_{s+1}, ..., p_n)$. The above correspondence provides a homeomorphism of $U_{\overline{w}}$ and $\mathrm{pr}_s(U') \times U_0 \times U''$, since close points in $U_{\overline{w}} \cap \partial\Delta$ come from close

curves $\gamma(t)$ which in turn bear close points in $\mathrm{pr}_s(U')\times(U_0\cap\partial\sigma)\times U''$. Consequently, we can introduce in U_0 coordinates $\theta_1,...,\theta_{n-s}$ such that

$$U_0=\{0\le\theta_1<\varepsilon,\ -\varepsilon<\theta_j<\varepsilon,\ j=2,...,n-s\},\quad U_0\cap\partial\sigma=\{\theta_1=0\}\ ,$$

and $M_{i_{s+1}...i_n}$, $(i_{s+1},...,i_n)\in\mathrm{pr}_{n-s}(\Lambda)$, are continuous functions of $\theta_1,...,\theta_{n-s}$, vanishing at zero. Thus, formulae (8) allow us to parameterize $\mathbb{C}Ch(F)\cap U_{\overline{w}}$ by the following subset in $\mathrm{pr}_s(U')\times U_0\times U''$:

$$x_1=x_1^0+G',\quad v_1=v_1^0+H'\ ,$$

where G' and H' are vanishing at zero smooth functions of $x_2,...,x_s,v_2,...,v_n,\theta_1,...,\theta_{n-s}$. This subset is a subvariety of codimension 2 with a boundary on $\mathrm{pr}_s(U')\times(U_0\cap\partial\sigma)\times U''$. Hence $\mathbb{C}Ch(F)\cap U_{\overline{w}}$ is a subvariety of codimension 2 in $U_{\overline{w}}$ with boundary on $U_{\overline{w}}\cap\partial\Delta$.

(v) The statement of Proposition for $\mathbb{R}Ch(F)$ can be proved as in the complex case. $\Box$

Corollary 1.6 *If* $\dim\Delta=2$, $\Delta\subset\mathbb{R}^2$, *then* $\mathbb{C}Ch(F)$ *is a smooth surface with boundary.*

1.4 Digression: real resolution of toric varieties

Let $\Delta\subset(\mathbb{R}_+^*)^n$ be an n-dimensional polytope, F be a non-degenerate polynomial with Newton polytope Δ. Denote by $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ the toric variety over $\mathbb{C}$ defined by Δ, and by $Z(F)$ the (closed) hypersurface in $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ defined by F.

Proposition 1.7 *There exists an equivariant surjective map* $\nu:\mathbb{C}\Delta\to\mathrm{Tor}_{\mathbb{C}}(\Delta)$ *such that*

- $\nu(\mathbb{C}Ch(F))=Z(F)$,
- $\nu\big|_{\mathbb{C}I(\Delta)}$ *takes* $\mathbb{C}I(\Delta)$ *diffeomorphically to* $(\mathbb{C}^*)^n\subset\mathrm{Tor}_{\mathbb{C}}(\Delta)$,
- *for any face* δ, $\dim\delta=s<n$, *the map* $\nu\big|_{\mathbb{C}I(\delta)}$ *is a submersion of* $\mathbb{C}I(\delta)$ *onto* $(\mathbb{C}^*)^s\subset\mathrm{Tor}_{\mathbb{C}}(\delta)\subset\mathrm{Tor}_{\mathbb{C}}(\Delta)$.

Proof. Let us represent $\mathrm{Tor}_{\mathbb{C}}(\Delta)$ as the closure of the variety

$$\{(z_1^{i_1}...z_n^{i_n})_{(i_1,...,i_n)\in\Delta\cap\mathbb{Z}^n}\ :\ (z_1,...,z_n)\in(\mathbb{C}^*)^n\}\subset\mathbb{C}P^{N-1}\ ,$$

where $N=\#(\Delta\cap\mathbb{Z}^n)$ (see, for example [12]). We define $\nu\big|_{\mathbb{C}I(\Delta)}=(\mathbb{C}\mu_\Delta)^{-1}$, which is a required diffeomorphism of $\mathbb{C}I(\Delta)$ and $(\mathbb{C}^*)^n\subset\mathrm{Tor}_{\mathbb{C}}(\Delta)$, and extend it to $\partial\mathbb{C}\Delta$. Namely, given a face $\delta\subset\Delta$ and a point $\overline{w}=\mathbb{C}\mu_\delta(\overline{z})\in\mathbb{C}I(\delta)$, $\overline{z}=(z_1,...,z_n)\in(\mathbb{C}^*)^n$, we put $\nu(\overline{w})=(a_{i_1...i_n})_{(i_1,...,i_n)\in\Delta\cap\mathbb{Z}^n}$, where

$$a_{i_1...i_n}=z_1^{i_1}...z_n^{i_n},\ (i_1,...,i_n)\in\delta,\quad a_{i_1...i_n}=0,\ (i_1,...,i_n)\in\Delta\backslash\delta\ .$$

Clearly, $\nu\big|_{\mathbb{C}I(\delta)}$ satisfies the required property. It remains only to explain that ν is continuous. Indeed, in the previous notation $\overline{w} = \lim_{t\to\infty} \mathbb{C}\mu_\Delta(\gamma(t))$, where

$$\gamma(t) = (z_1 t^{k_1} + O(t^{k_1 - 1}), ..., z_n t^{k_n} + O(t^{k_n - 1})), \quad t > 0 \ ,$$

is a curve in $(\mathbb{C}^*)^n$ with $(k_1, ..., k_n) \in \mathbb{Z}^n \backslash \{0\}$ belonging to the dual cone of δ with respect to Δ. Then one can easily check that $\lim_{t\to\infty} \gamma(t) \in \mathbb{C}P^{N-1}$ is just $\nu(\overline{w})$ defined as above. $\square$

1.5 Gluing of charts

Let $\mathcal{S}$ be a subdivision of a polytope $\Delta \subset \mathbb{R}^n$ into equidimensional polytopes: $\Delta = \Delta_1 \cup ... \cup \Delta_N$ (*i.e.*, $\Delta_i \cap \Delta_j$ is empty or a common proper face), and let $\mathcal{A} = \{A_{\overline{i}}, \ \overline{i} \in \Delta \cap \mathbb{Z}^n\}$ be a collection of complex numbers such that $A_{\overline{i}} \neq 0$ if $\overline{i}$ is a vertex of some Δ_i, $1 \leq i \leq N$. Further on, speaking on subdivisions of polytopes and corresponding collections of numbers, we always assume the above properties.

Assume that the polynomials

$$F_k(z_1, ..., z_n) = \sum_{(i_1, ..., i_n) \in \Delta_k} A_{i_1 ... i_n} z_1^{i_1} \cdot ... \cdot z_n^{i_n} \ , \quad k = 1, ..., N,$$

are non-degenerate. The union of the complex charts of the polynomials $F_1, ..., F_N$

$$\mathbb{C}Ch(\mathcal{S}, \mathcal{A}) = \bigcup_{k=1}^{N} \mathbb{C}Ch(F_k)$$

is called a *C-hypersurface in* $\mathbb{C}\Delta$. The union of the real charts of $F_1, ..., F_N$

$$\mathbb{R}Ch(\mathcal{S}, \mathcal{A}) = \bigcup_{k=1}^{N} \mathbb{R}Ch(F_k)$$

is called a *C-hypersurface in* $\mathbb{R}\Delta$.

Proposition 1.8 *Let $\Delta \subset (\mathbb{R}^*_+)^n$. Then the set $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ is a PL-manifold of codimension 2 in $\mathbb{C}\Delta$ with boundary $\partial\mathbb{C}Ch(\mathcal{S}, \mathcal{A}) = \partial\mathbb{C}\Delta \cap \mathbb{C}Ch(\mathcal{S}, \mathcal{A})$. If all the numbers in $\mathcal{A}$ are real, then $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ is invariant with respect to* Conj, *the set $\mathbb{R}Ch(\mathcal{S}, \mathcal{A})$ is a PL-manifold of codimension 1 in $\mathbb{R}\Delta$ with boundary $\partial\mathbb{R}Ch(\mathcal{S}, \mathcal{A}) = \partial\mathbb{R}\Delta \cap \mathbb{R}Ch(\mathcal{S}, \mathcal{A})$.*

Remark 1.9 *(a) Neither $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$, nor $\mathbb{R}Ch(\mathcal{S}, \mathcal{A})$ are smooth in general. Indeed, for the polynomials*

$$F_1(x, y) = x(y - x - 1), \quad F_2(x, y) = xy - x + 1$$

with the Newton triangles

$$\Delta_1 = [(1, 0), (1, 1), (2, 0)], \quad \Delta_2 = [(0, 0), (1, 0), (1, 1)]$$

the closures of curves

$$\mu_{\Delta_1}(\{F_1=0\})=\left\{\left(\frac{2+3x}{2+2x}\ ,\ \frac{x+1}{2+2x}\right)\ :\ x>0\right\},$$

$$\mu_{\Delta_2}(\{F_2=0\})=\left\{\left(\frac{2x-1}{2x}\ ,\ \frac{x-1}{2x}\right)\ :\ x>0\right\}$$

have the common limit point $(1,1/2)$ *on the edge* $[(1,0),(1,1)]$, *and different tangent lines*

$$y=\frac{1}{2},\quad y=x-\frac{1}{2}$$

at this point.

(b) Similarly to Proposition 1.4, the statement of Proposition 1.8 holds for Δ *intersecting coordinate hyperplanes, when substituting* $\mathbb{C}\Delta\backslash\mathrm{Sing}(\mathbb{C}\Delta)$ *and* $\mathbb{R}\Delta\backslash\mathrm{Sing}(\mathbb{C}\Delta)$ *for* $\mathbb{C}\Delta$ *and* $\mathbb{R}\Delta$, *respectively.*

Proof of Proposition 1.8. In view of Proposition 1.4, we have to study only the behavior of $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ of faces $\delta\subset\Delta_k\cap\Delta_l$, $k\ne l$, $\dim\delta=s>0$.

Let $I(\delta)\subset I(\Delta)$, $\delta=\Delta_1\cap...\cap\Delta_k$, $\delta\not\subset\Delta_{k+1}\cup...\cup\Delta_N$. Pick a point $w\in\mathbb{C}I(\delta)\cap\mathbb{C}Ch(\mathcal{S},\mathcal{A})$. Following the proof of Proposition 1.4, we impose assumptions of step (iv) and obtain that a neighborhood U_w of w in $\mathbb{C}\Delta$ is the union of neighborhoods $U_{w,1},...,U_{w,k}$ of w in $\mathbb{C}\Delta_1,...,\mathbb{C}\Delta_k$, respectively, so that $U_{w,m}$ is parameterized by $\mathrm{pr}_s(U'_m)\times U_{0,m}\times U''$, where $\mathrm{pr}_s(U'_1)=...=\mathrm{pr}_s(U'_k)$, U'' is common for all $m=1,...,k$, and $U_{0,m}$ is a neighborhood of the point $\mathrm{pr}_{n-s}(\delta)=(p_{s+1},...,p_n)$ in $\mathrm{pr}_{n-s}(\Delta_m)$, and, moreover, these parameterizations are compatible on the common faces of $\Delta_1,...,\Delta_k$. In turn, $\mathbb{C}Ch(\mathcal{S},\mathcal{A})\cap U_{w,m}$ is parameterized by

$$\prod_{j=2}^s\{|x_j-x_j^0|<\varepsilon\}\times U_{0,m}\times\prod_{j=2}^n\{|v_j-v_j^0|<\varepsilon\}\ ,$$

and by (4) these parameterizations are compatible along common faces of $\Delta_1,...,\Delta_k$. Hence $\mathbb{C}Ch(\mathcal{S},\mathcal{A})\cap U_w=\bigcup_m\mathbb{C}Ch(\mathcal{S},\mathcal{A})\cap U_{0,m}$ is parameterized by

$$\prod_{j=2}^s\{|x_j-x_j^0|<\varepsilon\}\times\bigcup_{m=1}^k U_{0,m}\times\prod_{j=2}^n\{|v_j-v_j^0|<\varepsilon\}\ ,$$

which is homeomorphic to a $(2n-2)$-ball.

Similarly one treats the case $\delta\subset\partial\Delta$ and the real case. $\Box$

Definition 1.10 *A homeomorphism of (resp., an isotopy in)* $\mathbb{C}\Delta$ *is called tame if for any face* δ *of* Δ *the restriction of this homeomorphism (resp., isotopy) to* $\mathbb{C}\delta$ *is a homeomorphism of (resp., an isotopy in)* $\mathbb{C}\delta$. *In addition, we call such objects equivariant if they commute with* Conj.

Remark 1.11 *For given* Δ *and* $\mathcal{S}$, *and different* $\mathcal{A},\mathcal{A}'\ :\ \Delta\cap\mathbb{Z}^n\to\mathbb{C}$, *the C-hypersurfaces* $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ *and* $\mathbb{C}Ch(\mathcal{S},\mathcal{A}')$ *are tame isotopic in* $\mathbb{C}\Delta$ *(not equivariantly, in general). Indeed, one can connect* $\mathcal{A}$ *and* $\mathcal{A}'$ *by a family* $\mathcal{A}_t$, $t\in[0,1]$ *such that the polynomials* $F_{k,t}$ *are non-degenerate for all* $k=1,...,N$, $t\in[0,1]$.

1.6 Projectivization of charts

Let d and n be positive integers. Denote by T_d^n the simplex in $\mathbb{R}^n$ with vertices $(0,0,...,0)$, $(d,0,...,0)$, ..., $(0,...,0,d)$. Observe that $\mathbb{C}T_d^n$ is homeomorphic to a closed ball in $\mathbb{C}^n$. The group $S^1 = \{|v| = 1\}$ acts freely on $\partial\mathbb{C}T_d^n$ by

$$v \in S^1,\ (z_1,...,z_n) \in \partial\mathbb{C}T_d^n \ \mapsto\ (z_1v,...,z_nv) \in \partial\mathbb{C}T_d^n\ .$$

Proposition 1.12 *The quotient $\mathbb{C}T_d^n/S^1$ is equivariantly homeomorphic to the projective space $\mathbb{C}P^n$ so that the faces of T_d^n naturally correspond to the coordinate planes in $\mathbb{C}P^n$.*

Proof. The extended moment map $\mathbb{C}\mu_{T_d^n}$ takes $\mathbb{C}^n$ diffeomorphically onto $\mathrm{Int}(\mathbb{C}T_d^n)$. Its inverse extends up to a surjective map $\nu_d^n : \mathbb{C}T_d^n \to \mathbb{C}P^n$ in the following way. Consider a point $(0,z_1,...,z_n) \in \mathbb{C}P^n \backslash \mathbb{C}^n$, the line $\gamma(t) = (z_1t,...,z_nt) \in \mathbb{C}^n$, $t \in (0,\infty)$, and $\lim_{t\to\infty} \mathbb{C}\mu_{T_d^n}(\gamma(t))$. Then $\lim_{t\to\infty} \mathbb{C}\mu_{T_d^n}(\gamma(t)) = \mathbb{C}\mu_\delta(z_1,...,z_n)$, where

$$\delta = \left\{\sum_{z_k\neq 0} i_k = d\right\} \cap \bigcap_{z_k=0}\{i_k = 0\}$$

is a face of T_d^n. Then we put $\nu_d^n(\mathbb{C}\mu_\delta(z_1,...,z_n) = (0,z_1,...,z_n) \in \mathbb{C}P^n$. One can easily verify that ν_d^n is well-defined, surjective, continuous, commutes with Conj, sends orbits of the S^1-action on $\partial\mathbb{C}T_d^n$ into the points of the hyperplane $\{z_0 = 0\}$ in $\mathbb{C}P^n$, and establishes a one-to-one correspondence between the faces of T_d^n and the coordinate planes in $\mathbb{C}P^n$. $\Box$

Remark 1.13 *Further on we always assume $\mathbb{C}T_d^n/S^1$ to be identified with $\mathbb{C}P^n$ via the map ν_d^n.*

Definition 1.14 *In the notation of section 1.5, given a subdivision $T_d^n = \Delta_1 \cup ... \cup \Delta_N$ and a collection of complex numbers $\mathcal{A} = \{A_{\bar{i}}\ :\ \bar{i} \in T_d^n \cap \mathbb{Z}^n\}$, define a C-hypersurface of degree d in $\mathbb{C}P^n$ as*

$$P\mathbb{C}Ch(\mathcal{S},\mathcal{A}) = \nu_d^n(\mathbb{C}Ch(\mathcal{S},\mathcal{A})) \subset \mathbb{C}P^n\ .$$

If the numbers $\mathcal{A} = \{A_{\bar{i}}\ :\ \bar{i} \in T_d^n \cap \mathbb{Z}^n\}$ are real, the C-hypersurface is called real and its real part $\nu_d^n(\mathbb{R}Ch(\mathcal{S},\mathcal{A}))$ is denoted by $P\mathbb{R}Ch(\mathcal{S},\mathcal{A})$.

Proposition 1.15 *If F is a non-degenerate polynomial of degree d in n variables, then $P\mathbb{C}Ch(F) \subset \mathbb{C}P^n$ coincides with the projective closure $P\{F = 0\}$ of the affine hypersurface $\{F = 0\}$.*

Proof. Straightforward from $\nu_d^n(\mathbb{C}Ch(F)) = P\{F = 0\}$ and the fact that $\mathbb{C}Ch(F) \cap \partial(\mathbb{C}T_d^n)$ is invariant with respect to the S^1-action. $\Box$

Definition 1.16 *A homeomorphism of (resp., an isotopy in) $\mathbb{C}P^n$ is called tame if its restriction to any coordinate plane is a homeomorphism of (resp., an isotopy in) that plane. In addition, we call such objects equivariant if they commute with* Conj*.*

Proposition 1.17 *In the previous notation, $P\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ is a PL-submanifold of codimension 2 in $\mathbb{C}P^n$. It is invariant with respect to* Conj *if all the numbers in $\mathcal{A}$ are real.*

Proof. First, we show that the C-hypersurface $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$ is a PL-manifold in $\mathbb{C}T^n_d$ of codimension 2 with boundary

$$\partial\mathbb{C}Ch(\mathcal{S},\mathcal{A}) = \mathbb{C}Ch(\mathcal{S},\mathcal{A}) \cap \partial\mathbb{C}T^n_d\ ,$$

and if all the numbers in $\mathcal{A}$ are real $\mathbb{C}Ch(\mathcal{S},\mathcal{A}$ is invariant with respect to Conj, and $\mathrm{Fix}(\mathrm{Conj}) = \mathbb{R}Ch(\mathcal{S},\mathcal{A})$ is a PL-manifold in $\mathbb{R}T^n_d$ of codimension 1 with boundary

$$\partial\mathbb{R}Ch(\mathcal{S},\mathcal{A}) = \mathbb{R}Ch(\mathcal{S},\mathcal{A}) \cap \partial\mathbb{R}T^n_d\ .$$

By Proposition 1.8 it is enough to consider only the union of charts along the coordinate hyperplanes. For any polytope $\Delta \subset \mathbb{R}^n$, put $\Delta_t = (t,...,t) + \Delta$, $t \geq 0$.

The family $\mathbb{C}\mu_{\Delta_{k,t}}$, $0 \leq t \leq 1$, connects $\mathbb{C}\mu_{\Delta_k}$ with $\mathbb{C}\mu_{\Delta_{k,1}}$, $k = 1,...,N$, and defines an isotopy of $\mathrm{Cl}(\mathbb{C}\mu_{\Delta_{k,t}}(F_k = 0))$, the closure of $\mathbb{C}\mu_{\Delta_{k,t}}(F_k = 0)$, $0 < t \leq 1$, which degenerates into $\mathbb{C}Ch(F_k)$, $k = 1,...,N$. The same holds for any proper face δ of $\Delta_1,...,\Delta_N$ and the corresponding truncation of $F_1,...,F_N$.

Since $\mathrm{Sing}(\mathbb{C}\Delta_{k,1}) = \emptyset$, $k = 1,...,N$, the set $\mathcal{M}_1 = \bigcup_{k=1}^N \mathbb{C}Ch(z_1...z_nF_k)$ is a PL-manifold with boundary on $\partial\mathbb{C}\widetilde{T}^n_d$, according to Proposition 1.4 (where $\widetilde{T}^n_d = (1,...,1) + T^n_d$). Denote some faces of $\widetilde{T}^n_d$ as follows:

$$\widetilde{T}^n_d(k) = \widetilde{T}^n_d \cap \{i_k = 1\}, \quad \widetilde{T}^n_d(k_1,...,k_s) = \bigcap_{j=1}^s \widetilde{T}^n_d(k_j)\ .$$

Put $\delta = \widetilde{T}^n_d(k)$. By (3)

$$\mathbb{C}\delta \cong \mathbb{C}T^{n-1}_d \times S^1. \tag{9}$$

Similarly, by (3) $\mathcal{M}_1 \cap \mathbb{C}\delta$ is a PL-manifold of dimension $2n-3$ with boundary on $\partial\mathbb{C}\delta$, satisfying

$$\mathcal{M}_1 \cap \mathbb{C}\delta \cong \left(\bigcup_{l=1}^N \mathbb{C}Ch(z_1...z_nF^\delta_l)\right) \times S^1, \tag{10}$$

where the product structure is compatible with that in (9). Using the product representations (9) and (10), we define

$$D^n(k) = \{(w_1,...,w_n)\ :\ (w_1,...,w_{k-1},w_{k+1},...,w_n) \in \mathbb{C}T^{n-1}_d,\ |w_k| \leq 1\}$$

$$\cong \mathbb{C}T_d^{n-1} \times D^2 \ ,$$

where D^2 is the closed 2-dimensional unit ball, and inside $D^n(k)$ the $(2n-2)$-dimensional manifold $M(k)$:

$$\{(w_1, ..., w_n) \ : \ (w_1, ..., w_{k-1}, 1, w_{k+1}, ..., w_n) \in \bigcup_{l=1}^{N} \mathbb{C}Ch(z_1...z_n F_l^\delta), \ |w_k| \leq 1\}$$

$$\cong \left(\bigcup_{l=1}^{N} \mathbb{C}Ch(\widetilde{F}_l^\delta) \right) \times D^2$$

with boundary on $\partial D^n(k)$ and such that $M(k) \cap \mathbb{C}\delta = \bigcup_{l=1}^{N} \mathbb{C}Ch(z_1...z_n F_l^\delta)$.

Similarly, given $\delta = \widetilde{T}_d^n(k_1, ..., k_s)$, we have by (3),

$$\mathbb{C}\delta \cong \mathbb{C}T_d^{n-s} \times (S^1)^s, \quad \mathcal{M}_1 \cap \mathbb{C}\delta \cong \left(\bigcup_{l=1}^{N} \mathbb{C}Ch(z_1...z_n F_l^\delta) \right) \times (S^1)^s,$$

and one defines

$$D^n(k_1, ..., k_s) = \{(w_1, ..., w_n) \ : \ (w_j)_{j \neq k_1, ..., k_s} \in \mathbb{C}T_d^{n-s}, \ |w_{k_1}|, ..., |w_{k_s}| \leq 1\}$$

$$\cong \mathbb{C}T_d^{n-s} \times (D^2)^s \ ,$$

and the $(2n-2)$-dimensional manifold $M(k_1, ..., k_s)$:

$$\{(w_1, ..., w_n) \ : \ (w_j)_{j \neq k_1, ..., k_s} \in \bigcup_{l=1}^{N} \mathbb{C}Ch(z_1...z_n F_l^\delta), \ |w_{k_1}|, ..., |w_{k_s}| \leq 1\}$$

$$\cong \left(\bigcup_{l=1}^{N} \mathbb{C}Ch(z_1...z_n F_l^\delta) \right) \times (D^2)^s$$

with boundary on $\partial D^n(k_1, ..., k_s)$ and such that

$$M(k_1, ..., k_s) \cap \mathbb{C}\delta = \bigcup_{l=1}^{N} \mathbb{C}Ch(z_1...z_n F_l^\delta) \ .$$

It can easily be shown that $\mathcal{T}_1 = \mathbb{C}\widetilde{T}_d^n \cup \bigcup_{s=1}^{n} \bigcup_{1 \leq k_1 < ... < k_s} D^n(k_1, ..., k_s)$ is the convex hull of $\mathbb{C}\widetilde{T}_d^n$ in $\mathbb{C}^n$, and $\mathcal{M}_1 \cup \bigcup_{s=1}^{n} \bigcup_{1 \leq k_1 < ... < k_s} M(k_1, ..., k_s)$ is a PL-manifold in $\mathcal{T}_1$ of codimension 2 with boundary on $\partial \mathcal{T}_1$.

Now, for any $t \in (0, 1)$, in the same way one constructs similar objects:

- complexifications of polytope

$$\mathbb{C}\Delta_{k,t} = \mathrm{Cl}(\mathbb{C}\mu_{\Delta_{k,t}}((\mathbb{C}^*)^n)), \quad \mathbb{C}T_d^n(t) = \bigcup_{k=1}^{N} \mathbb{C}\Delta_{k,t},$$

- union of charts $\mathcal{M}_t = \bigcup_{k=1}^{N} \mathrm{Cl}(\mathbb{C}\mu_{\Delta_k,t}(\{F_k = 0\}))$, which is a PL-manifold of codimension 2 in $\mathbb{C}T_d^n(t)$ with boundary on $\partial\mathbb{C}T_d^n(t)$,
- completion of $\mathbb{C}T_d^n(t)$ up to its convex hull

$$\mathcal{T}_t = \mathbb{C}T_d^n(t) \cup \bigcup_{s=1}^{n} \bigcup_{1\le k_1<...<k_s} D_t^n(k_1, ..., k_s),$$

where

$$D_t^n(k_1, ..., k_s) = \{(w_1, ..., w_n) \ : \ (w_j)_{j\neq k_1,...,k_s} \in \mathbb{C}T_d^{n-s}, \ |w_{k_1}|, ..., |w_{k_s}| \le t\}$$
$$\cong \mathbb{C}T_d^{n-s} \times (D^2(t))^s \ ,$$

$D^2(t)$ is a disc of radius t,

- a $(2n-2)$-dimensional manifold

$$M_t(k_1, ..., k_s) = \Big\{(w_1, ..., w_n) \ :$$
$$(w_j)_{j\neq k_1,...,k_s} \in \bigcup_{l=1}^{N} \mathrm{Cl}(\mathbb{C}\mu_{\delta_t}(\{F_l^\delta = 0\})), \ |w_{k_1}|, ..., |w_{k_s}| \le t\Big\}$$
$$\cong \left(\bigcup_{l=1}^{N} \mathrm{Cl}(\mathbb{C}\mu_{\delta_t}(\{F_l^\delta = 0\}))\right) \times (D^2(t))^s$$

- a PL-manifold

$$\mathcal{M}_t \cup \bigcup_{s=1}^{n} \bigcup_{1\le k_1<...<k_s} M_t(k_1, ..., k_s) \tag{11}$$

of codimension 2 in $\mathcal{T}_t$ with boundary on $\partial\mathcal{T}_t$.

If t varies from 1 to 0, $\mathcal{T}_t$ contracts from $\mathcal{T}_1$ to $\mathbb{C}T_d^n$ and the manifold (11) naturally contracts into the C-hypersurface $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ with boundary on $\partial\mathbb{C}T^n$.

To complete the proof of Proposition 1.17 we have to verify only that $\partial\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ is invariant with respect to the S^1-action on $\partial\mathbb{C}T_d^n$, and the quotient $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})/S^1$ is a closed manifold.

First, note that $\partial\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ is invariant with respect to the S^1-action. Indeed, $\partial\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ consists of charts of homogeneous polynomials which are invariant with respect to the S^1-action on $\partial\mathbb{C}T_d^n$. Second, for any edge of T_d^n there exists a combination of an automorphism of $\mathbb{Z}^n$ and shifts which puts this edge on a coordinate axis and the adjacent faces of T_d^n on the corresponding coordinate planes. Since such a transformation is compatible with ν_d^n defined via the extended moment map, the question on the behavior of $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ on $\partial\mathbb{C}T_d^n$ is reduced to that on coordinate planes, which has been treated above. $\square$

Remark 1.18 *Similarly to Remark 1.11, two C-hypersurfaces in $\mathbb{C}P^n$ of the same degree and with the same subdivision $\mathcal{S}$ of T_d^n are tame isotopic in $\mathbb{C}P^n$ (not equivariantly, in general).*

2 Topology of complex C-hypersurfaces

2.1 Complex version of the Viro theorem

Theorem 2.1 (Affine complex Viro theorem, see [30]). *In the notation of section 1.5, if $\Delta \subset (\mathbb{R}_+^*)^n$ and a subdivision $\mathcal{S}\ :\ \Delta = \Delta_1 \cup ... \cup \Delta_N$ is defined by a convex piecewise-linear function $\nu : T_d^n \to \mathbb{R}$, then $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ is tame isotopic in $\mathbb{C}\Delta$ (equivariantly, if all the numbers in $\mathcal{A}$ are real) to $\mathbb{C}Ch(F)$, where*

$$F(z_1, ..., z_n, t) = \sum_{(i_1,...,i_n)\in\Delta} A_{i_1...i_n} t^{\nu(i_1,...,i_n)} z_1^{i_1} \cdot ... \cdot z_n^{i_n}$$

and $t = \mathrm{const} > 0$ is sufficiently small.

Proof. Without loss of generality one can suppose that $\dim\Delta = n$, and ν is positive integral-valued at integral points. Consider the polynomial $\hat{F}(z_1, ..., z_n, t) = (1+t)F(z_1, ..., z_n, t)$ as a polynomial in $t, x_1, ..., x_n$, and denote by $\hat{\Delta}$ its Newton polytope in $\mathbb{R}^{n+1}$. The graph of ν is the "lower" part of $\partial\hat{\Delta}$, naturally projected onto $\Delta \subset \mathbb{R}^n = \{i_{n+1} = 0\} \subset \mathbb{R}^{n+1}$. Denote by $\hat{\Delta}_k$ the part of the graph of ν projected onto Δ_k, $k = 1, ..., N$. Denote by B the part of $\hat{\Delta}$ projected onto $\partial\Delta$. Clearly, B can be viewed as $\partial\Delta \times [0, 1]$.

Step 1. Introduce the set

$$H_0 = \{(w_1, ..., w_n, \nu(|w_1|, ..., |w_n|))\ :\ (|w_1|, ..., |w_n|) \in \Delta\} = \bigcup_{k=1}^{N} \hat{\Delta}_k \times (S^1)^n \subset \partial\mathbb{C}\hat{\Delta}\ ,$$

and the family of hypersurfaces $H_a = \mathrm{Cl}(\mathbb{C}\mu_{\hat{\Delta}}(\{t = a\})) \subset \mathbb{C}\hat{\Delta}$, $a \in (0, \varepsilon)$. We will show that the family H_a, $a \in [0, \varepsilon)$, is an isotopy in $\mathbb{C}\hat{\Delta}$.

The map $\mathbb{C}\mu_{\hat{\Delta}}$ takes $\{t = a\} \subset (\mathbb{C}^*)^{n+1}$ diffeomorphically into $\mathbb{C}I(\hat{\Delta})$. Considering $\lim_{\tau\to 0}\mu_{\hat{\Delta}}(\gamma(\tau))$ for all curves

$$\gamma(\tau) = \left(\lambda_1 x_1^{k_1}, ..., \lambda_n x_n^{k_n}, a\right) \subset (\mathbb{R}_+^*)^{n+1}, \quad \tau > 0,$$

with $\lambda_1, ..., \lambda_n > 0$, $(k_1, ..., k_n) \neq 0$, one can show (as in the proof of Proposition 1.4) that H_a being the closure of $\mathbb{C}\mu_{\hat{\Delta}}(\{t = a\})$ is a PL-manifold with boundary $\partial H_a = H_a \cap B$, which projects onto $\partial\Delta$. Moreover, $H_a \cap H_b = \emptyset$, $a \neq b$. The projection on the first n coordinates provides a homeomorphism of H_a, $a > 0$, and $\mathbb{C}\Delta$. Indeed, in $\mathbb{C}I(\hat{\Delta})$ this fact is reduced to the claim that

$$(x_1, ..., x_n) \in (\mathbb{R}_+^*)^n \mapsto \frac{\sum_{(i_1,...,i_n)\in\Delta} x_1^{i_1} ... x_n^{i_n} a^{\nu(i_1,...,i_n)} \cdot (i_1, ..., i_n)}{\sum_{(i_1,...,i_n)\in\Delta} x_1^{i_1} ... x_n^{i_n} a^{\nu(i_1,...,i_n)}} \in I(\Delta)$$

is a homeomorphism, which holds since the above map is a kind of the moment map [1, 2]. Similarly one shows that the projection is one-to-one on $\partial H_a \subset B$, taking into account that the limit of $\mu_{\hat{\Delta}}$ on a face of B is the moment map of this face. To

finish the proof one should note that $\bigcup_{a>0} H_a$ fill the intersection of $\mathbb{C}I(\hat{\Delta})$ with the space $\mathrm{Im}(w_{n+1})=0$, where $w_1,...,w_{n+1}$ are coordinates in $(\mathbb{C}^*)^{n+1}\supset\mathbb{C}\hat{\Delta}$.

Step 2. Now we show that $\mathbb{C}Ch(\hat{F})\cap H_a$, $a\in[0,\varepsilon)$, is an isotopy in $\mathbb{C}\hat{\Delta}$. For $\varepsilon>0$ small enough, $\{\hat{F}=0\}\cap\{t=a\}=\{F=0\}\cap\{t=a\}$ and the intersection is transverse in $(\mathbb{C}^*)^{n+1}$. This implies that the family $\mathbb{C}Ch(\hat{F})\cap H_a$, $a\in(0,\varepsilon)$, is an isotopy in $\mathbb{C}\hat{\Delta}$.

Now we pick a point $\overline{w}=(y_1v_1^0,...,y_nv_n^0,y_{n+1})\in H_0\cap\mathbb{C}Ch(\hat{F})$, where $(y_1,...,y_n)\in I(\delta)$ for some s-dimensional face of $\Delta_1,...,\Delta_N$, $y_{n+1}=\nu(y_1,...,y_n)$, and $(v_1^0,...,v_n^0)\in(S^1)^n$. Following the proof of Proposition 1.4 and using transformations

$$(x_1,...,x_n,t)\mapsto\left(t^{\beta_1}\prod_{k=1}^n x_k^{\alpha_{k1}},...,t^{\beta_n}\prod_{k=1}^n x_k^{\alpha_{kn}},t\right)$$

with $A=(\alpha_{kj})\in SL(n,\mathbb{Z})$, we can get δ to be parallel to the coordinate s-plane $\{i_1=...=0\}$, $\nu\big|_\delta=\nu_0$, and $\nu\big|_{\Delta\backslash\delta}>\nu_0$. Then we introduce a parameterization of a neighborhood $U_{\overline{w}}$ of the point $\overline{w}$ in $\mathbb{C}\hat{\Delta}\cap\{\mathrm{Im}(w_{n+1})=0\}$ by $\mathrm{pr}_s(U')\times U_0\times U''$ via an extension of the map $\mathbb{C}\mu_{\hat{\Delta}}$ such that, in the notation of the proof of Proposition 1.4, $\mathrm{pr}_s(U')$ is given by the first s inequalities in (7), U'' is given by (6), and U_0 is a neighborhood of the point $\mathrm{pr}_{n-s+1}(\delta)$ in $\mathrm{pr}_{n-s+1}(\hat{\Delta})$. Here $\mathrm{Int}(U_0)$ is identified with the domain $\mathrm{pr}_{n-s+1}(U')\subset(\mathbb{R}^*_+)^{n-s+1}$, given in the coordinates $x_{s+1},...,x_n,t$ by the relations

$$M_{i_{s+1},...,i_n,i_{n+1}}=x_{s+1}^{i_{s+1}}...x_n^{i_n}t^{i_{n+1}}<\varepsilon,\quad(i_{s+1},...,i_{n+1})\in\mathrm{pr}_{n-s+1}(\hat{\Delta}\backslash\Delta)\cap\mathbb{Z}^{n-s+1}\ ,$$

and $\mathbb{C}Ch(\hat{F})\cap\mathrm{Int}(U_{\overline{w}})$ is parameterized in $\mathrm{pr}_s(U')\times\mathrm{Int}(U_0)\times U''$ by equations (8), where G and H are vanishing at zero smooth functions of $x_2,...,x_s$, $v_1,...,v_n$ and $M_{i_{s+1},...,i_n,i_{n+1}}$, $(i_{s+1},...,i_{n+1})\in\mathrm{pr}_{n-s+1}(\hat{\Delta}\backslash\Delta)\cap\mathbb{Z}^{n-s+1}$.

Denote $\widetilde{\Delta}=\delta\times\mathrm{pr}_{n-s+1}(\hat{\Delta})$. Clearly, $\hat{\Delta}$ and $\widetilde{\Delta}$ coincide in a neighborhood of $\overline{w}$. Moreover, The maps $\mathbb{C}\mu_{\hat{\Delta}}$ and $\mathbb{C}\mu_{\widetilde{\Delta}}$ are connected by an isotopy on the domain $\mathrm{pr}_s(U')\times\mathrm{Int}(U_0)\times U''$: such an isotopy can be written explicitly by supplying the non-common summands in the formulae for $\mathbb{C}\mu_{\hat{\Delta}}$ and $\mathbb{C}\mu_{\widetilde{\Delta}}$ by a parameter running over $[0,1]$. This isotopy extends up to equivariant tame isotopy on $\mathrm{pr}_s(U')\times U_0\times U''$, so that replacing $\mathbb{C}\mu_{\hat{\Delta}}$ by $\mathbb{C}\mu_{\widetilde{\Delta}}$, one replace $U_{\overline{w}}$ by another neighborhood $\widetilde{U}_{\overline{w}}$ of $\overline{w}$ in $\mathbb{C}\widetilde{\Delta}\cap\{\mathrm{Im}(w_{n+1})\}$. Note that the map $\mathbb{C}\mu_{\widetilde{\Delta}}$ splits on $\mathrm{pr}_s(U')\times U_0\times U''$ into the product of

$$\mu_\delta:\mathrm{pr}_s(U')\to\delta,\quad\mu_{\mathrm{pr}_{n-s+1}(\hat{\Delta})}\to\mathrm{pr}_{n-s+1}(\hat{\Delta}),\quad\mathrm{Id}:U''\to U''.$$

Together with the result of Step 1 this allows us to introduce in U_0 coordinates $t,\theta_1,...,\theta_{n-s}$ so that

$$U_0=\{0\le t<\varepsilon,\ -\varepsilon<\theta_j<\varepsilon\},\quad U_0\cap\partial(\mathrm{pr}_{n-s+1}(\hat{\Delta}))=\{t=0\}\ ,$$

and $\mathbb{C}\mu_{\widetilde{\Delta}}(H_a)=\mathrm{pr}_s(U')\times\{t=a\}\times U''$. In addition, $\mu_{\mathrm{pr}_{n-s+1}(\hat{\Delta})}$ expresses $M_{i_{s+1},...,i_n,i_{n+1}}$, $(i_{s+1},...,i_{n+1})\in\mathrm{pr}_{n-s+1}(\hat{\Delta}\backslash\Delta)\cap\mathbb{Z}^{n-s+1}$, as continuous functions of

$t, \theta_1, ..., \theta_{n-s}$, vanishing at zero. Hence the closure of $\mathbb{C}\mu_{\widetilde{\Delta}}(\{F=0\})$ in $\widetilde{U}_{\overline{w}}$ is given by equations

$$x_1 = x_1^0 + G', \quad v_1 = v_1^0 + H', \tag{12}$$

where G' and H' are vanishing at zero smooth functions of $x_2, ..., x_s,\ v_2, ..., v_n$, $t, \theta_1, ..., \theta_{n-s}$. The variety (12) intersects any hypersurface $\mathrm{pr}_s(U') \times \{t=a\} \times U''$, $a \in [0, \varepsilon)$, transversally in $\mathrm{pr}_s(U') \times U_0 \times U'' \simeq \widetilde{U}_{\overline{w}} \simeq U_{\overline{w}}$, thereby proving that $\mathbb{C}Ch(\hat{F}) \cap H_a$, $a \in [0, \varepsilon)$, is an isotopy in $\mathbb{C}\hat{\Delta}$.

Step 3. For $a \neq 0$, the projection of $\mathbb{C}Ch(\hat{F}) \cap H_a$ into $\mathbb{C}\Delta$ is the closure of the image of $\{F=0\} \cap \{t=a\} \subset (\mathbb{C}^*)^{n+1}$ by the map

$$\mathbb{C}\mu_{\Delta,a}(\overline{x}, \overline{v}) = \frac{\sum_{(i_1,...,i_n)\in\Delta}(i_1 v_1, ..., i_n v_n) \cdot \left(x_1^{i_1} ... x_n^{i_n} \sum_{(i_1,...,i_n,k)\in\hat{\Delta}} a^k \right)}{\sum_{(i_1,...,i_n)\in\Delta} \left(x_1^{i_1} ... x_n^{i_n} \sum_{(i_1,...,i_n,k)\in\hat{\Delta}} a^k \right)} . \tag{13}$$

The map $\mathbb{C}\mu_{\Delta,a}$ is connected with $\mathbb{C}\mu_\Delta$ on $\{t=a\} \simeq (\mathbb{C}^*)^n$ by the isotopy

$$\frac{\sum_{(i_1,...,i_n)\in\Delta}(i_1 v_1, ..., i_n v_n) \cdot \left(x_1^{i_1} ... x_n^{i_n} \left(\sum_{(i_1,...,i_n,k)\in\hat{\Delta}} a^k(1-\tau) + \tau \right) \right)}{\sum_{(i_1,...,i_n)\in\Delta} \left(x_1^{i_1} ... x_n^{i_n} \left(\sum_{(i_1,...,i_n,k)\in\hat{\Delta}} a^k(1-\tau) + \tau \right) \right)} , \tag{14}$$

providing a tame isotopy of the projection of $\mathbb{C}Ch(\hat{F}) \cap H_a$ with $\mathbb{C}Ch(F\big|_{t=a}) \subset \mathbb{C}\Delta$. Similarly, the projection of $\mathbb{C}Ch(\hat{F}) \cap \mathbb{C}\hat{\Delta}_k \cap \{\mathrm{Im}(w_{n+1}) = 0\}$ into $\mathbb{C}\Delta_k$, $1 \leq k \leq N$, is the closure of the image of $\{F^{\hat{\Delta}_k} = 0\} \cap \{t=1\}$ by the map $\mathbb{C}\mu_{\hat{\Delta}_k}$ which coincides with the closure of the image of $\{F_k = 0\}$ by the map $\mathbb{C}\mu_{\Delta_k}$, *i.e.*, $\mathbb{C}Ch(F_k)$.

So, Theorem is proven in the complex case. If all the numbers in $\mathcal{A}$ are real, then the isotopy constructed is equivariant. $\square$

Theorem 2.2 (Projective complex Viro theorem, see [30]). *Let, in the notation of Theorem 2.1, $\Delta = T_d^n$ and the subdivision $\mathcal{S}$ be defined by a convex piecewise-linear function $\nu : T_d^n \to \mathbb{R}$. Then $P\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ is tame isotopic in $\mathbb{C}P^n$ (equivariantly, if all the numbers in $\mathcal{A}$ are real) to $P\mathbb{C}Ch(F)$, where*

$$F(z_1, ..., z_n, t) = \sum_{i_1+...+i_n \leq d} A_{i_1...i_n} t^{\nu(i_1,...,i_n)} z_1^{i_1} \cdot ... \cdot z_n^{i_n} = 0,$$

and $t = \mathrm{const} > 0$ is sufficiently small.

Proof. We multiply all the polynomials by $z_1...z_n$ and apply Theorem 2.1 to the simplex $\widetilde{T_d^n} = T_d^n + (1, ..., 1)$ and correspondingly shifted $\mathcal{S}, \mathcal{A}$. Then we note that the isotopy of $\mathbb{C}Ch(z_1...z_n F)$ and $\bigcup_{k=1}^N \mathbb{C}Ch(z_1...z_n F_k)$ in $\mathbb{C}\widetilde{T_d^n}$ is tame and compatible with the action of $(S^1)^n$ on $\partial\mathbb{C}\widetilde{T_d^n}$. This allows us to take quotient by this action as was done in the proof of Proposition 1.17 and obtain the required isotopy. $\square$

Proposition 2.3 *Let two subdivisions $\mathcal{S} = \{\Delta_k,\ k = 1, ..., N\}$ and $\mathcal{S}' = \{\Delta_{kl},\ l = 1, ..., r_k,\ k = 1, ..., N\}$ of a polytope $\Delta \subset (\mathbb{R}_+^*)^n$ satisfy*

$$\Delta_k = \bigcup_{l=1}^{r_k} \Delta_{kl}, \quad k = 1, ..., N,$$

so that the subdivision $\mathcal{S}'$ is given by piecewise-linear function $\nu : \Delta \to \mathbb{R}$, whose restrictions $\nu_k = \nu\big|_{\Delta_k}$, $k = 1, ..., N$, are convex. Then the varieties $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ and $\mathbb{C}Ch(\mathcal{S}', \mathcal{A}')$ are tame isotopic, provided $\mathcal{A}, \mathcal{A}' : \Delta \to \mathbb{C}$ define non-degenerate polynomials $F_k,\ F_{kl},\ l = 1, ..., r_k,\ k = 1, ..., N$. Similarly, given two subdivisions $\mathcal{S},\ \mathcal{S}'$ of T_d^n satisfying the previous assumptions, the C-hypersurfaces of degree d constructed out of $\mathcal{S}, \mathcal{A}$ and $\mathcal{S}', \mathcal{A}'$ are tame isotopic in $\mathbb{C}P^n$.

Proof. Consider the case $\Delta \subset (\mathbb{R}_+^*)^n$. Fix $a > 0$. Let the maps $\mathbb{C}\mu_{\Delta_k, a} : (\mathbb{C}^*)^n \to \mathbb{C}\Delta_k$, $k = 1, ..., N$, be defined by (13), where summations run over Δ_k and $\hat{\Delta}_k$, constructed as in the proof of Theorem 2.1 by means of ν_k. Put

$$F_{k,a} = \sum_{(i_1,...,i_n)\in\Delta_k} A'_{i_1...i_n} z_1^{i_1} ... z_n^{i_n} a^{\nu(i_1,...,i_n)}, \quad k = 1, ..., N.$$

As shown in the proof of Theorem 2.1, the closure $\mathbb{C}Ch_a(F_{k,a})$ of $\mathbb{C}\mu_{\Delta_k,a}(\{F_{k,a} = 0\})$ in $\mathbb{C}\Delta_k$ is tame isotopic to $\bigcup_l \mathbb{C}Ch(F_{kl})$. Moreover, for any face $\delta = \Delta_k \cap \Delta_j$, the isotopies in $\mathbb{C}\Delta_k$ and $\mathbb{C}\Delta_l$ coincide on $\mathbb{C}\delta$. Then, for each $k = 1, ..., N$, we connect $\mathbb{C}\mu_{\Delta_k,a}$ with $\mathbb{C}\mu_{\Delta_k}$ by an isotopy (14), thereby obtaining the required isotopy of $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ and $\mathbb{C}Ch(\mathcal{S}', \mathcal{A}')$.

The same argument proves the statement for $\Delta = T_d^n$. $\Box$

A subdivision $\Delta = \Delta_1 \cup ... \cup \Delta_N$ is called *maximal* if it cannot be refined. In this case all the integral points in Δ are vertices of $\Delta_1, ..., \Delta_N$, and these polytopes are simplices.

Corollary 2.4 *Given a polytope $\Delta \subset (\mathbb{R}_+^*)^n$, any C-hypersurface in $\mathbb{C}\Delta$ is tame isotopic to a C-hypersurface constructed out of a maximal subdivision of Δ. The same is true for C-hypersurfaces in $\mathbb{C}P^n$.*

Proof. Let $\Delta = \Delta_1 \cup ... \cup \Delta_N$, and $f : \Delta \to \mathbb{R}$ be a smooth convex function. Define a piecewise-linear function $\nu : \Delta \to \mathbb{R}$ as follows: put $\nu(\bar{i}) = f(\bar{i})$, $\bar{i} \in \Delta \cap \mathbb{Z}^n$, then define the graph of $\nu\big|_{\Delta_k}$ to be the "lower" part of the convex hull of $\{(\bar{i}, \nu(\bar{i}) : \bar{i} \in \Delta_k\}$. The function ν defines a maximal subdivision of Δ inscribed into the initial subdivision and satisfying the conditions of Proposition 2.3, which completes the proof. $\Box$

Corollary 2.5 *Any C-curve in $\mathbb{C}P^2$ is tame isotopic (not equivariantly, in general) to an algebraic curve of the same degree.*

Proof. By Corollary 2.4 we can assume that a C-curve in $\mathbb{C}P^2$ is constructed out of a maximal triangulation of T_d^2. We will transform any given triangulation into a convex triangulation, so that in each transformation step the conditions of Proposition 2.3 hold true.

Let $\mathcal{S}$ be a triangulation of T_d^2, $O(\mathcal{S})$ denote the star of the origin O with respect to this triangulation, and outside $O(\mathcal{S})$ the triangulation $\mathcal{S}$ is maximal. We construct a triangulation $\mathcal{S}'$ of T_d^2 such that $O(\mathcal{S}') \supset O(\mathcal{S})$ and $O(\mathcal{S}') \neq O(\mathcal{S})$. Then, in finitely many steps we come to $O(\mathcal{S}) = T_d^2$, which corresponds to a convex triangulation, and the required statement will follow from Theorem 2.2 and Proposition 2.3.

Let $O, P_1, ..., P_r$ be all the vertices of $O(\mathcal{S})$ numbered successively clockwise along $\partial O(\mathcal{S})$. Any segment $[P_i, P_{i+1}]$ either lies on ∂T_d^2, or is an edge of a unique triangle $T_i \in \mathcal{S}$, $T_i \not\subset O(\mathcal{S})$.

(i) Assume that, for some $i = 1, ..., r$, the vertex $Q \neq P_i, P_{i+1}$ of T_i lies between the straight lines(OP_i) and (OP_{i+1}), and $Q \notin (OP_i) \cup (OP_{i+1})$ (see Figure 2a). Then we change the subdivision of T_d^2 as shown in Figure 2b,c. These changes satisfy the conditions of Proposition 2.3 and lead to a triangulation with a strongly greater star of O.

(ii) Assume that, for some $i = 1, ..., r-1$, $T_i = T_{i+1}$, *i.e.*, the vertices of the latest triangle are P_i, P_{i+1}, P_{i+2} (see Figure 2d). Then we perform the transformation shown in Figure 2e, once again increasing the star of O.

(iii) Assume that there are no triangles T_i as in (i), (ii). Then any triangle T_i is either "left", *i.e.*, the vertex Q lies on (OP_i) or above (OP_i), or "right", *i.e.*, Q lies on (OP_{i+1}) or below (OP_{i+1}). If there exist "left" triangles, consider the "left" triangle T_i with the minimal i. If $i = 1$, we have the situation shown in Figure 2f. Then we change triangulation as shown in Figure 2g, increasing the star of O. If $i > 1$, then the triangle T_{i-1} must be "right", which means that we have a situation shown in Figure 2h. Then we change triangulation as shown in Figure 2i, increasing the star of O. $\square$

2.2 Basic properties of C-hypersurfaces

The real part of a real C-hypersurface M in $\mathbb{C}P^n$ (see Definition 1.14) is denoted by $\mathbb{R}M$.

Proposition 2.6 *A complex C-hypersurface M of degree d in $\mathbb{C}P^n$ is an orientable manifold, homologous to an algebraic hypersurface of degree d in $\mathbb{C}P^n$. If M is real its real part $\mathbb{R}M$ is a closed manifold, mod 2 homologous to the real point set of a real algebraic hypersurface of degree d in $\mathbb{R}P^n$.*

Proof. The Jacobian of the moment map $\mu_\Delta : (\mathbb{R}_+^*)^n \to \text{Int}(\Delta)$, $\mu_\Delta(\overline{x}) = (\mu_\Delta^{(1)}(\overline{x}), ..., \mu_\Delta^{(n)}(\overline{x}))$, is positive. Hence the Jacobian of the extended moment map $\mathbb{C}\mu_\Delta : (\mathbb{C}^*)^n \to \mathbb{C}I(\Delta)$ is positive in the coordinates

$$x_1 = |z_1|,\ v_1 = \frac{z_1}{|z_1|},\ \dots\ ,\ x_n = |z_n|,\ v_n \frac{z_n}{|z_n|}\ ,$$

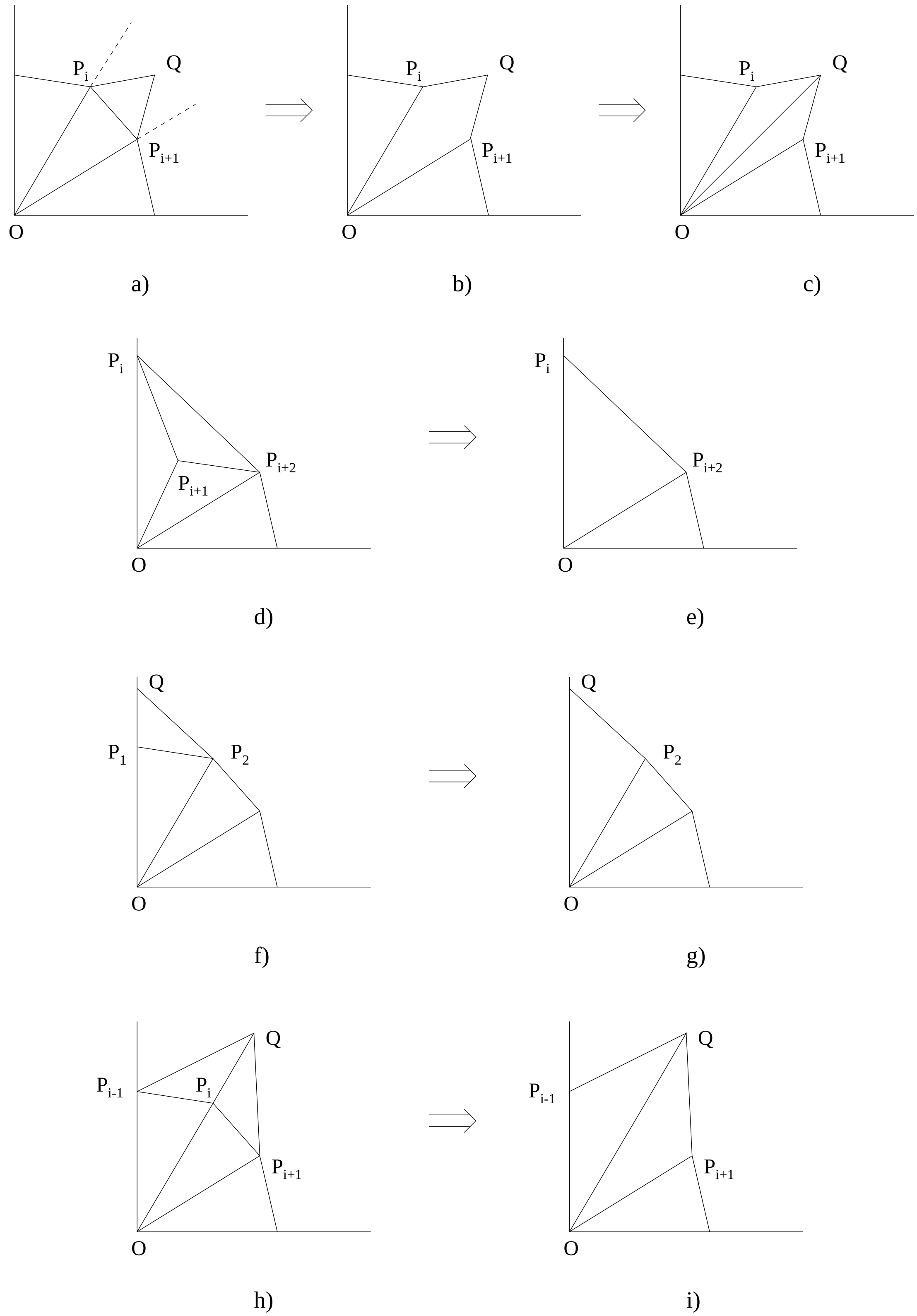

Figure 2: Transformation of a triangulation

because this is a diffeomorphism, and at a point with $v_1 = ... = v_n = 1$ one can easily compute

$$\det\left(\frac{D(\mathbb{C}\mu_\Delta)}{D(x_1, v_1, ..., x_n, v_n)}\right) = \det\left(\frac{D(\mu_\Delta)}{D(x_1, ..., x_n)}\right) \cdot \prod_{j=1}^{n} \mu_\Delta^{(j)}(x_1, ..., x_n) > 0 \ .$$

This means, in particular, that $\mathbb{C}\mu_\Delta$ canonically defines an orientation of images of complex subvarieties of $(\mathbb{C}^*)^n$. Therefore, the open subset

$$\bigcup_{k=1}^{N} (\mathbb{C}Ch(F_k) \cap \mathbb{C}I(\Delta_k))$$

of $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ is canonically orientable. So, to complete the proof of orientability, one should verify that these orientations are compatible when gluing the charts $\mathbb{C}Ch(F_k)$, $\mathbb{C}Ch(F_j)$ with $\Delta_k \cap \Delta_j = \delta$ being a common facet (the gluing along faces of lower dimensions does not affect the orientation). In other words, the orientations of $\mathbb{C}Ch(F_k^\delta) = \mathbb{C}Ch(F_j^\delta)$ induced by $\mathbb{C}Ch(F_k)$ and $\mathbb{C}Ch(F_j)$ are opposite.

Without loss of generality suppose that δ is contained in a hyperplane $i_1 = a > 0$, $\Delta_k \subset \{i_1 \le a\}$, and $\Delta_j \subset \{i_1 \ge a\}$. Then by construction, $\mathbb{C}\Delta_k$ induces on $\mathbb{C}\delta$ an orientation defined by the form

$$-dv_1 \wedge dx_2 \wedge dv_2 \wedge ... \wedge dx_n \wedge dv_n,$$

and $\mathbb{C}\Delta_j$ induces on $\mathbb{C}\delta$ the opposite orientation defined by

$$dv_1 \wedge dx_2 \wedge dv_2 \wedge ... \wedge dx_n \wedge dv_n.$$

On the other hand, a coorienting 2-vector bundle on $\mathbb{C}Ch(F_k) \cap \mathbb{C}I(\Delta_k)$ can continuously be extended to a coorienting 2-bundle on $\mathbb{C}Ch(F_k^\delta) \subset \mathbb{C}I(\delta)$, and the same for Δ_j. Indeed, let the complex straight line Λ

$$z_2 = ... = z_n = xv, \quad z_1 = x_0 = \text{const}, \quad x \in (0, \infty), \ |v| = 1,$$

with small $z_0 > 0$ meet the hypersurfaces $\{F_k = 0\}$, $\{F_k^\delta = 0\}$ transversally. Then the family of surfaces Σ_λ, $\lambda \in [0, 1]$,

$$\left\{ \frac{\sum_{(i_1,...,i_n)\in\Delta_k} x^{i_2+...+i_n} x_0^{i_1} \lambda^{a-i_1} \cdot (i_1, i_2 v, ..., i_n v)}{\sum_{(i_1,...,i_n)\in\Delta_k} x^{i_2+...+i_n} x_0^{i_1} \lambda^{a-i_1}} \ : \ x \in (0, \infty), \ |v| = 1 \right\}$$

is a diffeotopy, connecting $\Sigma_0 = \mathbb{C}\mu_\delta(\Lambda)$, which coorients $\mathbb{C}Ch(F_k^\delta)$ in $\mathbb{C}I(\delta)$, and $\Sigma_1 = \mathbb{C}\mu_{\Delta_k}(\Lambda)$, which coorients $\mathbb{C}Ch(F_k)$ in $\mathbb{C}I(\Delta_k)$.

Comparing this with the previous remark on orientations of $\mathbb{C}\delta$, one completes the proof of the orientability of a complex C-hypersurface.

At last, $[\mathbb{C}Ch(\mathcal{S}, \mathcal{A})] \in H_{2n-2}(\mathbb{C}P^n)$ and $[\mathbb{R}Ch(\mathcal{S}, \mathcal{A})] \in H_{n-1}(\mathbb{R}P^n, \mathbb{Z}/2)$ can easily be computed by induction considering the intersection of the complex and real charts with the coordinate planes. $\Box$

Proposition 2.7 *Any (real) C-hypersurface M is (equivariantly) tame isotopic to a close smooth manifold M_{sm} of codimension 2 in $\mathbb{C}P^n$.*

Proof. Let $M = \mathbb{C}Ch(\mathcal{S},\mathcal{A})/S^1$, where $\mathcal{S} = \{\Delta_1, ..., \Delta_N\}$, $\mathcal{A} = \{A_{\overline{i}} : \overline{i} \in T^n_d \cap \mathbb{Z}^n\}$. Put $F_m = \sum_{\overline{i}\in\Delta_m} A_{\overline{i}}\overline{z}^{\overline{i}}$, $m = 1, ..., N$. We construct two nonvanishing $\mathbb{R}$-linearly independent sections $\overline{s}$ and $\overline{s}'$ of the bundle $T\mathbb{C}P^n\big|_{U(M)}$, where $U(M)$ is a neighborhood of M in $\mathbb{C}P^n$, such that $\overline{s}$ is equivariant, $\overline{s}'$ is anti-equivariant, and the 2-subbundle $\mathrm{Span}_{\mathbb{R}}\{\overline{s},\overline{s}'\} \subset T\mathbb{C}P^n\big|_M$ is transverse to $T\mathbb{C}Ch(F_m)$ in $T\mathbb{C}P^n\big|_{\mathbb{C}Ch(F_m)}$ for any $m = 1, ..., N$. This will imply the existence of a smooth $2(n-1)$-manifold M_{sm} isotopic (equivariantly, if $\mathrm{Conj}(M) = M$) to M, close to M and transverse to the 2-bundle $\mathrm{Span}_{\mathbb{R}}\{\overline{s},\overline{s}'\}$. Namely, one smoothes M, pushing it along the trajectories of the vector field $\overline{s}$ in a neighborhood of $\bigcup_{m=1}^N \partial\mathbb{C}\Delta_m$.

(i) First, we shift T^n_d into $\widetilde{T}^n_d = (1, ..., 1) + T^n_d$ (shifting respectively $\mathcal{S},\mathcal{A}$ as well), and construct sections $\overline{s}$ and $\overline{s}'$ of $T\mathbb{C}^n$ defined in a neighborhood of $\mathbb{C}Ch(\mathcal{S},\mathcal{A})$, with the above properties and an additional one: invariance with respect to the S^1-action on $\partial\mathbb{C}\widetilde{T}^n_d$. The latter allows us to obtain the required sections of $T\mathbb{C}P^n$ along the procedure described in the proof of Proposition 1.17.

(ii) Fix $m = 1, ..., N$ and consider the hypersurface $\widetilde{F}_m(\overline{z}) = F_m(e^{z_1}, ..., e^{z_n}) = 0$ in $\mathbb{C}^n$. Define vector fields $\overline{s}_1$ and $\overline{s}'_1$ on $\mathbb{C}^n$ by

$$\overline{s}_1(\overline{z}) = \frac{\mathrm{Conj}(\mathrm{grad}\widetilde{F}_m)}{|\mathrm{Conj}(\mathrm{grad}\widetilde{F}_m)|}, \quad \overline{s}'_1(\overline{z}) = \frac{\mathrm{Conj}(\mathrm{grad}\widetilde{F}_m)\sqrt{-1}}{|\mathrm{Conj}(\mathrm{grad}\widetilde{F}_m)\sqrt{-1}|} .$$

They do not vanish along $\{\widetilde{F}_m = 0\}$ and span a 2-bundle orthogonal to $T\{\widetilde{F}_m = 0\}$ in $T\mathbb{C}^n\big|_{\{\widetilde{F}_m=0\}}$. These vector fields are $2\pi\sqrt{-1}$-periodic in each coordinate $z_1, ..., z_n$, and their normalized images $\overline{s}$ and $\overline{s}'$ by the differential $D(\mathbb{C}\mu_{\Delta_m} \circ \exp)$ of the map $\mathbb{C}\mu_{\Delta_m} \circ \exp : \mathbb{C}^n \to \mathbb{C}\Delta_m$ give a 2-bundle $\mathrm{Span}_{\mathbb{R}}\{\overline{s},\overline{s}'\}$ on $\mathbb{C}I(\Delta_m) \cap \mathbb{C}Ch(F_m)$ transverse to the tangent bundle $T(\mathbb{C}I(\Delta_m) \cap \mathbb{C}Ch(F_m))$. We will show that these vector fields continuously extend to $\bigcup_{m=1}^N(\partial\mathbb{C}\Delta_m) \cap U(M)$.

(iii) Let δ be a face of Δ_m, $\overline{w}$ be a point in $\mathbb{C}Ch(F_m) \cap \mathbb{C}I(\delta)$, and $U_{\overline{w}} \subset \mathbb{C}\Delta_m$ be a neighborhood of $\overline{w}$ as introduced in the step (iv) of the proof of Proposition 1.4. We claim that, for any point $\overline{z} \in \mathbb{C}^n$ such that $\mathbb{C}\mu_\Delta(e^{\overline{z}}) \in U_{\overline{w}} \cap \mathbb{C}I(\Delta_m)$, one has

$$\left|\left|\frac{D(\mathbb{C}\mu_{\Delta_m} \circ \exp)_{\overline{z}}}{||D(\mathbb{C}\mu_{\Delta_m} \circ \exp)_{\overline{z}}||} - \frac{D(\mathbb{C}\mu_\delta \circ \exp)_{\overline{\alpha}}}{||D(\mathbb{C}\mu_\delta \circ \exp)_{\overline{\alpha}}||}\right|\right| < c\varepsilon , \tag{15}$$

and

$$\left|\frac{\mathrm{grad}(F_m \circ \exp)}{|\mathrm{grad}(F_m \circ \exp)|}(\overline{z}) - \frac{\mathrm{grad}(F^\delta_m \circ \exp)}{|\mathrm{grad}(F^\delta_m \circ \exp)|}(\overline{\alpha})\right| < c\varepsilon , \tag{16}$$

where $\overline{\alpha} \in \mathbb{C}^n$ is some point, satisfying $\mathbb{C}\mu_\Delta(\exp(\overline{\alpha})) = \overline{w}$, and $c > 0$ depends only on the coefficients of F and on the point $\overline{w}$.

(iv) To show (15), note that $\mathbb{C}\mu_{\Delta_m} \circ \exp$ splits into $\mu_{\Delta_m} \circ \exp : \mathbb{R}^n \to I(\Delta_m)$ and $\exp : \mathbb{R}^n\sqrt{-1} \to (S^1)^n$. Inequality (15) for the differential of $\exp : \mathbb{R}^n\sqrt{-1} \to (S^1)^n$

instead of $\mathbb{C}\mu_{\Delta_m} \circ \exp$ immediately follows from (6). Then, acting by a transformation from $SL(n,\mathbb{Z})$ and a shift, we move Δ_m, δ into $\widetilde{\Delta}_m, \widetilde{\delta}$ such that $\widetilde{\delta}$ lies in a coordinate s-plane. According to (7),

$$|e^{(\overline{i},\mathrm{Re}\overline{z})} - e^{(\overline{i},\mathrm{Re}\overline{\alpha})}| < c_1\varepsilon,\ \overline{i} \in \widetilde{\delta} \cap \mathbb{Z}^n, \quad e^{(\overline{i},\mathrm{Re}\overline{z})} < c_1\varepsilon,\ \overline{i} \in (\widetilde{\Delta}_m \backslash \widetilde{\delta}) \cap \mathbb{Z}^n,$$

where $c_1 > 0$ depends only on Δ_m, thereby this implies (15), because $D(\mu_{\widetilde{\Delta}_m} \circ \exp)$ is represented by the matrix

$$\left(\frac{\sum_{\overline{i}\in\widetilde{\Delta}_m} i_p i_q e^{(\overline{i},\mathrm{Re}\overline{z})} \cdot \sum_{\overline{i}\in\widetilde{\Delta}_m} e^{(\overline{i},\mathrm{Re}\overline{z})} - \sum_{\overline{i}\in\widetilde{\Delta}_m} i_p e^{(\overline{i},\mathrm{Re}\overline{z})} \cdot \sum_{\overline{i}\in\widetilde{\Delta}_m} i_q e^{(\overline{i},\mathrm{Re}\overline{z})}}{(\sum_{\overline{i}\in\widetilde{\Delta}_m} e^{(\overline{i},\mathrm{Re}\overline{z})})^2} \right)_{p,q=1,\dots,n},$$

and $D(\mu_{\widetilde{\delta}} \circ \exp)$ does not vanish.

(v) Similarly, $F_m(e^{\overline{z}}) = e^{(\overline{i}_0,\overline{z})} \sum_{\overline{i}\in\Delta} A_{\overline{i}} e^{(\overline{i},\overline{z})}$, where by (6) and (7)

$$|e^{(\overline{i},\overline{z})} - e^{(\overline{i},\overline{\alpha})}| < c_2\varepsilon,\ \overline{i} \in \delta \cap \mathbb{Z}^n, \quad |e^{(\overline{i},\overline{z})}| < c_2\varepsilon,\ \overline{i} \in (\Delta \backslash \delta) \cap \mathbb{Z}^n,$$

where $f_2 > 0$ depends on Δ_m. So, we obtain

$$\mathrm{grad}(F_m \circ \exp)(\overline{z}) = e^{(\overline{i}_0,\overline{z})} \left(\sum_{\overline{i}\in\delta} A_{\overline{i}} \overline{i} e^{(\overline{i},\overline{z})} + \sum_{\overline{i}\in\Delta\backslash\delta} A_{\overline{i}} \overline{i} e^{(\overline{i},\overline{z})} \right),$$

which immediately implies (16), because $\sum_{\overline{i}\in\delta} A_{\overline{i}} \overline{i} e^{(\overline{i},\overline{\alpha})} = \mathrm{grad}(F_m^\delta \circ \exp)(\overline{\alpha}) \neq 0$.

(vi) Relations (15) and (16) provide continuous extension of the vector fields $\overline{s}$ and $\overline{s}'$ on $\bigcup_{m=1}^N (\partial\mathbb{C}\Delta_m) \cap U(M)$ so that they remain $\mathbb{R}$-linearly independent and belong to $T\mathbb{C}I(\delta)$ along $\mathbb{C}I(\delta)$ for any proper face δ of Δ_m, $m = 1, ..., N$. Moreover, the restrictions of $\overline{s}$ and $\overline{s}'$ on $\mathbb{C}I(\delta)$ depend only on δ and F_m^δ, common for all $\Delta_m \supset \delta$, hence they are compatible with S^1-action on $\partial\mathbb{C}\widetilde{T}_d^n$, and we are done. $\Box$

Proposition 2.8 *Given a (real) C-hypersurface M, the tangent bundle to its smoothing M_{sm} is (equivariantly) isotopic to a (equivariant) bundle of complex hyperplanes. In particular, M_{sm} possesses an (equivariant) almost complex structure. If M is real, the above isotopy has fixed intersection with $T\mathbb{R}P^n\big|_{M_{sm}}$ (equal to $T\mathbb{R}M_{sm}$). Furthermore, the complex structure in $\mathbb{C}P^n$ can be (equivariantly) deformed into an almost complex structure, compatible with the metric and for which TM_{sm} is invariant.*

Proof. We construct the required isotopy in few steps.

Step 1. In the proof of Proposition 2.7 we have constructed sections $\overline{s}$, $\overline{s}'$ of the bundle $T\mathbb{C}P^n\big|_{M_{sm}}$, which are linearly independent at any point $\overline{w} \in M_{sm}$ and such that the 2-bundle $\mathrm{Span}_{\mathbb{R}}\{\overline{s}, \overline{s}'\}$ is transversal to TM_{sm}. Suppose that $\mathbb{C}P^n$ is equipped with a Hermitian metric compatible with the complex structure and the complex conjugation. Then there exists a (linear) isotopy of the bundle TM_{sm} into the $(2n-2)$-bundle $(\mathrm{Span}_{\mathbb{R}}\{s, s'\})^\perp$.

Step 2. Assume that $M = P\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$, where $\mathcal{S}$ is a subdivision $T_d^n = \Delta_1 \cup ... \cup \Delta_N$. Via the map $\nu_d^n : \mathbb{C}T_d^n \to \mathbb{C}P^n$ we pull back $\overline{s}$ and $\overline{s}'$ to sections of the bundle $T\mathbb{C}T_d^n$ restricted to $(\nu_d^n)^{-1}(M_{sm})$ (which will be denoted by M_{sm} for abuse of notations), as well as pull back the complex structure J and the Hermitian metric. Note also that $T\mathbb{C}T_d^n \simeq \mathbb{C}T_d^n \times \mathbb{C}^n$ possesses the standard complex structure being just the multiplication by $\sqrt{-1}$.

All this data on $\mathbb{C}T_d^n$ is invariant with respect to the S^1-action on $\partial\mathbb{C}T_d^n$, as well as the procedures used further; hence the isotopies we construct in $T\mathbb{C}T_d^n$ can be pushed to $T\mathbb{C}P^n$.

The following lemma implies that there exists an isotopy of the bundle $(\mathrm{Span}_{\mathbb{R}}\{\overline{s}, \overline{s}'\})^{\perp}$ into the bundle $\left(\mathrm{Span}_{\mathbb{R}}\{\overline{s}, \overline{s}\sqrt{-1}\}\right)^{\perp}$.

Lemma 2.9 *In the above notation, for any $\overline{w} \in M_{sm}$, no vector $(1-\lambda)\overline{s}'(\overline{w}) + \lambda\overline{s}\sqrt{-1}$, $0 \le \lambda \le 1$, is $\mathbb{R}$-proportional to $\overline{s}(\overline{w})$.*

Proof. Let $\overline{w} \in \mathbb{C}\Delta_m$, $1 \le m \le N$. We have

$$\overline{s}(\overline{w}) = D\widetilde{\mathbb{C}\mu}_{\Delta_m}(\overline{b}), \quad \overline{s}'(\overline{w}) = D\widetilde{\mathbb{C}\mu}_{\Delta_m}(\overline{b}\sqrt{-1}),$$

where $\widetilde{\mathbb{C}\mu}_{\Delta_m} : \mathbb{C}^n \to \mathbb{C}^n$ is defined by

$$\mathbb{C}^n = \mathbb{R}^n \oplus \mathbb{R}^n\sqrt{-1} \overset{\log\widetilde{\mu}_{\Delta_m} \oplus \mathrm{Id}}{\longrightarrow} \mathbb{R}^n \oplus \mathbb{R}^n\sqrt{-1} = \mathbb{C}^n \ , \tag{17}$$

$$\log\widetilde{\mu}_{\Delta_m} = (\log\mu_{1,\Delta_m}, ..., \log\mu_{n,\Delta_m}), \quad \mu_{\Delta_m} = (\mu_{1,\Delta_m}, ..., \mu_{n,\Delta_m}) \ ,$$

and $\overline{b}$ is a nonzero vector on $\mathbb{C}^n$.

Assume that

$$\kappa\overline{s}(\overline{w}) = (1-\lambda)\overline{s}'(\overline{w}) + \lambda\overline{s}(\overline{w})\sqrt{-1} \tag{18}$$

for some $\kappa \in \mathbb{R}$, $\lambda \in (0,1)$ and $\overline{w} \in \mathbb{C}I(\Delta_m)$. Splitting $\overline{b}$ into $\overline{b}_r + \overline{b}_i\sqrt{-1}$ with $\overline{b}_r, \overline{b}_i \in \mathbb{R}^n$, we obtain in view of (17)

$$\begin{aligned} \overline{s}(\overline{w}) &= D(\log\mu_{\Delta_m})(\overline{b}_r) + \overline{b}_i\sqrt{-1}, \\ \overline{s}(\overline{w})\sqrt{-1} &= -\overline{b}_i + D(\log\mu_{\Delta_m})(\overline{b}_r)\sqrt{-1}, \\ \overline{s}'(\overline{w}) &= -D(\log\mu_{\Delta_m})(\overline{b}_i) + \overline{b}_r\sqrt{-1} \ . \end{aligned}$$

Suppose that $\kappa = 0$. Setting the expressions for $\overline{s}(\overline{w})\sqrt{-1}$ and $\overline{s}'(\overline{w})$ into (18) we get that the operator $D(\log\mu_{\Delta_m})$ has a negative eigenvalue, what is impossible. Indeed, the matrix of $D(\log\mu_{\Delta_m})$ is a product of

$$A = \left(\frac{\sum_{\overline{i}\in\Delta} i_p i_q e^{(\overline{i},\overline{x})} \cdot \sum_{\overline{i}\in\Delta} e^{(\overline{i},\overline{x})} - (\sum_{\overline{i}\in\Delta} i_p e^{(\overline{i},\overline{x})})(\sum_{\overline{i}\in\Delta} i_q e^{(\overline{i},\overline{x})})}{(\sum_{\overline{i}\in\Delta} e^{(\overline{i},\overline{x})})^2}\right)_{p,q=1,\dots,n} ,$$

$$B = \mathrm{diag}\left(\frac{\sum_{\overline{i}\in\Delta} i_p e^{(\overline{i},\overline{x})}}{\sum_{\overline{i}\in\Delta} e^{(\overline{i},\overline{x})}}\right)_{p=1,\dots,n} .$$

Here both A and B are positive definite, for instance,

$$(A\overline{b},\overline{b})=\frac{\sum_{\overline{i},\overline{j}\in\Delta_m}e^{(\overline{i},\overline{x})}e^{(\overline{j},\overline{x})}(\sum_{p=1}^n(i_p-j_p)b_p)^2}{(\sum_{\overline{i}\in\Delta_m}e^{(\overline{i},\overline{x})})^2}>0\ ,$$

hence AB rotates any vector by an angle $<\pi$, so cannot have negative eigenvalues.

Suppose that $\kappa\ne 0$. Plugging the above expressions for $\overline{s}(\overline{w})$, $\overline{s}(\overline{w})\sqrt{-1}$ and $\overline{s}'(\overline{w})$ into (18), we obtain that the operator

$$(\lambda-\lambda^2)\mathrm{Id}+(1-2\lambda+2\lambda^2+\kappa^2)D(\log\mu_{\Delta_m})+(\lambda-\lambda^2)D(\log\mu_{\Delta_m})^2$$

vanishes at $b_r\ne 0$. Both the roots of the polynomial

$$\varphi(X)=\lambda-\lambda^2+(1-2\lambda+2\lambda^2+\kappa^2)X+(\lambda-\lambda^2)X^2$$

are negative; hence $D(\log\mu_{\Delta_m})$ should have a negative eigenvalue in contrary to the previous argument, and we are done.

The same argument proves the required statement when $\overline{w}\in\partial\mathbb{C}\Delta_m$ in view of (15) and (16). $\square$

Step 3. Now we claim that there exists an isotopy of the bundle $\big(\mathrm{Span}_{\mathbb{R}}\{\overline{s},\overline{s}\sqrt{-1}\}\big)^{\perp}$ into the bundle $(\mathrm{Span}_{\mathbb{R}}\{\overline{s},J\overline{s}\})^{\perp}$ which is a bundle of complex hyperplanes. We use the fact that, for any $\overline{w}\in\mathbb{C}T^n_d$ and $\lambda\in(0,1)$, no vector $\lambda\overline{s}(\overline{w})\sqrt{-1}+(1-\lambda)J\overline{s}(\overline{w})$ is $\mathbb{R}$-proportional to $\overline{s}(\overline{w})$, which follows from Lemma 2.9 applied to the case $\Delta_m=T^n_d$.

Step 4. If M is real then the initial 2-bundle $\mathrm{Span}_{\mathbb{R}}\{\overline{s},\overline{s}'\}$ and all the constructions are Conj-invariant; hence the isotopies and the resulting bundle of complex hyperplanes are equivariant. To satisfy the condition that the isotopy has fixed intersection with $T\mathbb{R}P^n\big|_{M_{sm}}$ equal to $T\mathbb{R}M_{sm}$, in the very beginning we choose a hermitian metric on $\mathbb{C}P^n$, compatible with the complex structure and such that the vector field $\overline{s}\big|_{\mathbb{R}M_{sm}}\subset T\mathbb{R}P^n$ is orthogonal to $T\mathbb{R}M_{sm}$.

Step 5. Let us extend the above constructed isotopy to a neighborhood U of M_{sm}. Take a smooth function $\rho:\mathbb{C}P^n\to[0,1]$ (equivariant in the real case), which is 0 in $\mathbb{C}P^n\backslash U$ and 1 on M_{sm}. We have three $(2n-2)$-bundles on U:

$$L^{(0)}=(\mathrm{Span}\{\overline{s},\overline{s}'\})^{\perp},\quad L^{(1/2)}=\big(\mathrm{Span}\{\overline{s},\overline{s}\sqrt{-1}\}\big)^{\perp},\quad L^{(1)}=(\mathrm{Span}\{\overline{s},J\overline{s}\})^{\perp}\ .$$

Let $L^{(t)}$ be an isotopy of $L^{(0)}$ to $L^{(1)}$ via $L^{(1/2)}$. By constructions in Steps 2, 3, there exist families of isometries

$$Q_\lambda:T\mathbb{C}P^n\big|_U\to T\mathbb{C}P^n\big|_U,\ 0\le\lambda\le\frac{1}{2},\quad Q_\lambda(L^{(\lambda)})=L^{(1/2)},\quad Q_{1/2}=\mathrm{Id},$$

$$R_\lambda:T\mathbb{C}P^n\big|_U\to T\mathbb{C}P^n\big|_U,\ \frac{1}{2}\le\lambda\le 1,\quad R_\lambda(L^{(\lambda)})=L^{(1)},\quad R_1=\mathrm{Id}\ .$$

Then we define a deformation J_t, $t\in[0,1]$, of the complex structure $J=J_0$ by

$$J_t=R_{1-t\rho(\overline{w})}J(R_{1-t\rho(\overline{w})})^{-1},\quad t\rho(\overline{w})\le\frac{1}{2},$$

$$J_t=Q_{1-t\rho(\overline{w})}R_{1/2}J(Q_{1-t\rho(\overline{w})}R_{1/2})^{-1},\quad t\rho(\overline{w})>\frac{1}{2}\ .\quad\square$$

2.3 Algebraic covering

Introduce the map $\Pi_m : \mathbb{C}P^n \to \mathbb{C}P^n$, $[z_0 : \ldots : z_n] \mapsto [z_0^m : \ldots : z_n^m]$.

Proposition 2.10 *For any C-hypersurface M of degree d in $\mathbb{C}P^n$ there exists a number $m_0 > 0$ such that for any integer $m > m_0$ the preimage $\Pi_m^{-1}(M)$ of M is a PL-manifold tame isotopic to a smooth algebraic hypersurface of degree md in $\mathbb{C}P^n$.*

Proof. Let M be defined by polynomials $F_i(z_1, ..., z_n)$, $i = 1, ..., N$, with Newton polytopes $\Delta_1, ..., \Delta_N$, where $\Delta_1 \cup ... \cup \Delta_N = T_d^n$. The polynomials

$$\widetilde{F}_i(z_1, ..., z_n) = F_i(z_1^m, ..., z_n^m), \quad i = 1, ..., N,$$

have the Newton polytopes $m\Delta_i$, $i = 1, ..., N$, and define a C-hypersurface $\widetilde{M}$ of degree md.

Lemma 2.11 *The manifold $\widetilde{M}$ is tame isotopic in $\mathbb{C}P^n$ to $\Pi_m^{-1}(M)$.*

Proof. Let $\Delta \subset \mathbb{R}^n$ be a polytope of dimension n. Let us show that the diffeomorphism $\mu_\Delta \Pi_m \mu_{m\Delta}^{-1} : I(m\Delta) \to I(\Delta)$, where $\Pi_m : (\mathbb{R}_+)^n \to (\mathbb{R}_+)^n$ acts as $(x_1, ..., x_n) \mapsto (x_1^m, ..., x_n^m)$, extends to a homeomorphism $\varphi : m\Delta \to \Delta$ such that, for any proper face $\sigma \subset \Delta$, the map $\varphi|_{m\sigma} : m\sigma \to \sigma$ is a homeomorphism depending only on σ (and not on Δ). Indeed, since Π_m commutes with $SL(n, \mathbb{Z})$ acting on $(\mathbb{R}_+)^n$, we can replace Δ by its image under some $g \in SL(n, \mathbb{Z})$ such that the given face σ will lie on the coordinate hyperplane $\{i_1 = 0,\ i_2 \cdot ... \cdot i_n \neq 0\}$. Then $\mu_{t(\Delta)}$ and $\mu_{mt(\Delta)}$ extend by the formulas similar to (1) to homeomorphisms of $(\mathbb{R}_+)^n \cup \{x_1 = 0,\ x_2 \cdot ... \cdot x_n \neq 0\}$ and $I(t(\Delta)) \cup I(t(\sigma))$ and $I(mt(\Delta)) \cup I(mt(\sigma))$, respectively. This gives us a homeomorphism $I(m\Delta) \cup I(m\sigma) \to I(\Delta) \cup I(\sigma)$, which does not depend on the choice of t, and we are done.

Consider now the following commutative diagram:

$$\begin{array}{ccccc}
\coprod_{1\le k\le N}(\mathbb{C}^*)^n & \xrightarrow{\coprod_{1\le k\le N}\mathbb{C}\mu_{m\Delta_k}} & \mathbb{C}T_{md}^n & \xrightarrow{\mathbb{C}\mu} & \mathbb{C}P^n \\
\Big\downarrow \coprod_{1\le k\le N}\Pi_m & & \Big\downarrow \varphi_m & & \Big\downarrow \widetilde{\Pi}_m \\
\coprod_{1\le k\le N}(\mathbb{C}^*)^n & \xrightarrow{\coprod_{1\le k\le N}\mathbb{C}\mu_{\Delta_k}} & \mathbb{C}T_d^n & \xrightarrow{\mathbb{C}\mu} & \mathbb{C}P^n
\end{array}$$

where $\coprod_{1\le k\le N}(\mathbb{C}^*)^n$ means the disjoint union of N copies of $(\mathbb{C}^*)^n$. The map φ_m is defined on any $\mathbb{C}\Delta_k$ as follows:

$$(x_1, ..., x_n) \in \Delta_k \mapsto \mu_{\Delta_k}\Pi_m\mu_{m\Delta_k}^{-1}(x_1, ..., x_n) \in \Delta_k,$$

$$(v_1, ..., v_n) \in (S^1)^n \mapsto (v_1^m, ..., v_n^m) \in (S^1)^n\ ,$$

with $\mu_{\Delta_k}\Pi_m\mu_{m\Delta_k}^{-1}$ extended on the whole Δ_k. This definition is correct since the extensions coming from Δ_k and Δ_j with a common face are the same on the common face, as shown above. The map $\widetilde{\Pi}_m$ is defined by this diagram. Let us show that $\widetilde{\Pi}_m$

is tame isotopic to Π_m. First, $\Pi_m(x_0v_0,...,x_nv_n) = (x_0^mv_0^m,...,x_n^mv_n^m)$ is tame isotopic to $\pi'_m(x_0v_0,...,x_nv_n) = (x_0v_0^m,...,x_nv_n^m)$. On the other hand, the homeomorphism

$$\coprod_{1\le k\le N} \mu_{\Delta_k}\Pi_m\mu^{-1}_{m\Delta_k} : T^n_{md} \to T^n_d$$

in the definition of φ_m is tame isotopic to the homothety, turning φ_m into

$$\varphi'_m(x_1v_1,...,x_nv_n) = \frac{1}{m}(x_1v_1^m,...,x_nv_n^m)\ ,$$

and completing the proof of Lemma, since $\pi'_m = \mathbb{C}\mu\circ\varphi'_m\circ(\mathbb{C}\mu)^{-1}$, and $\varphi_m^{-1}(\mathbb{C}Ch(F_k)) = \mathbb{C}Ch(F_k(z_1^m,...,z_n^m))$. $\square$

Lemma 2.12 *There exists m_0 such that, for any $m\ge m_0$, the subdivision $T^n_{md} = m\Delta_1\cup...\cup m\Delta_N$ admits a convex refinement.*

Proof. Let Γ be the graph of a smooth convex function of n variables, say $f(i_1,...,i_n) = i_1^2+...+i_n^2$, and let $\mathrm{pr}:\Gamma\to\mathbb{R}^n$ be the projection. Denote by $\mathrm{sk}^{n-1}(\Delta)$ the $(n-1)$-skeleton of the subdivision $\Delta_1\cup...\cup\Delta_N$. Clearly, $\mathrm{pr}^{-1}(\mathrm{sk}^{n-1}(\Delta))$ lies on the boundary of its convex hull. The same is true for

$$\mathrm{pr}^{-1}\left(\mathrm{sk}^{n-1}(\Delta)\cap\frac{1}{m}\mathbb{Z}^n\right). \tag{19}$$

If m is big enough, one can define a required refinement by the piecewise-linear convex function whose graph is the lower part of the boundary of the convex hull of the set (19). $\square$

Now to finish the proof of Proposition 2.10 it remains to apply Theorem 2.2 and Proposition 2.3. $\square$

Denote by $\chi(X)$ the Euler characteristic of X, by $\mathrm{sign}(X)$ the signature of X if X is a manifold whose dimension is divisible by 4, and by χ^n_d (resp., sign^n_d, if n is odd) the Euler characteristic (resp., the signature) of a nonsingular algebraic hypersurface of degree d in $\mathbb{C}P^n$.

Corollary 2.13 *Any C-hypersurface M of degree d in $\mathbb{C}P^n$ satisfies*

$$\chi(M) = \chi^n_d,\quad \mathrm{sign}(M) = \mathrm{sign}^n_d\ .$$

Proof.

(i) The equality $\chi(M) = \chi^n_d$ follows immediately by induction from Proposition 2.10 and the behavior of the Euler characteristic under ramified coverings.

(ii) Suppose that n is odd and consider a nonsingular algebraic hypersurface M' of degree d in $\mathbb{C}P^n$. Let m_0 be as in Proposition 2.10. Take any prime number $p > m_0$. Note that $\widetilde{M'} = \Pi_p^{-1}(M')\subset\mathbb{C}P^n$ is an algebraic hypersurface of degree

pd tame isotopic to $\widetilde{M}=\Pi_p^{-1}(M)$. The deck transformation group of the coverings $\Pi_p:\widetilde{M}\to M$ and $\Pi_p:\widetilde{M}'\to M'$ is $G\simeq(\mathbb{Z}/p)^n$. According to [17],

$$\mathrm{sign}(M)=\frac{1}{|G|}\sum_{g\in G}\mathrm{sign}(g,\widetilde{M}),\quad \mathrm{sign}(M')=\frac{1}{|G|}\sum_{g\in G}\mathrm{sign}(g,\widetilde{M}')\ . \tag{20}$$

For $g=\mathrm{Id}\in G$, we have $\mathrm{sign}(g,\widetilde{M})=\mathrm{sign}(\widetilde{M})=\mathrm{sign}(\widetilde{M}')=\mathrm{sign}(g,\widetilde{M}')$. Pick $g\ne\mathrm{Id}$. The twisted signatures $\mathrm{sign}(g,\widetilde{M})$ and $\mathrm{sign}(g,\widetilde{M}')$ depend on the embedding of $\mathrm{Fix}(g)$ in $\widetilde{M}$ and $\widetilde{M}'$, respectively, and the action of g in the tangent and normal bundles of $Y=\mathrm{Fix}(g)\subset\widetilde{M}$ and $Y'=\mathrm{Fix}(g)\subset\widetilde{M}'$ (see Theorem 6.12 [3]). We do not compute the twisted signatures along Theorem 6.12 [3], but use Hirzebruch's formula for the signature of ramified coverings. Namely, let $\langle g\rangle\subset G$ be the cyclic group of order p generated by g. According to [16],

$$p\cdot\mathrm{sign}(\widetilde{M}/\langle g\rangle)=\sum_{i=0}^{p-1}\mathrm{sign}(g^i,\widetilde{M})$$

is a universal function of p and the signatures of self-intersections $Y\circ Y$, $Y\circ Y\circ Y$, $\dots$. Note that $Y=\mathrm{Fix}(g)\subset\widetilde{M}$ and $Y'=\mathrm{Fix}(g)\subset\widetilde{M}'$ are the intersections of $\widetilde{M}$ and $\widetilde{M}'$, respectively, with the same collection of coordinate planes. Hence, due to the tame isotopy of $\widetilde{M}$ and $\widetilde{M}'$, one has $\mathrm{sign}(Y\circ Y)=\mathrm{sign}(Y'\circ Y')$, $\mathrm{sign}(Y\circ Y\circ Y)=\mathrm{sign}(Y'\circ Y'\circ Y')$, and so on. Therefore for any $g\in G$ we have

$$\sum_{i=0}^{p-1}\mathrm{sign}(g^i,\widetilde{M})=\sum_{i=0}^{p-1}\mathrm{sign}(g^i,\widetilde{M}')\ ,$$

which immediately implies the required equality of the right hand sides in (20), since $G\simeq(\mathbb{Z}/p)^n$ can be decomposed in the union of cyclic subgroups such that the only intersection of any two of these subgroups is $\mathrm{Id}\in G$. $\Box$

Corollary 2.14 *Let M be a C-hypersurface of degree d in $\mathbb{C}P^n$. Then*

- *M is simply connected if $n>2$,*
- *$\pi_1(\mathbb{C}P^n\backslash M)=\mathbb{Z}/d\mathbb{Z}$ if $n\ge 2$.*

Proof.

(i) To show that M is simply connected for $n>2$, note that an algebraic hypersurface of dimension greater than 1 is simply connected. Let $m>m_0$ be as in Proposition 2.10. Consider a loop γ in M. We can move it slightly so that it does not meet the coordinate hyperplanes in $\mathbb{C}P^n$. Since $\Pi_m^{-1}(M)$ is simply connected, $m^n\cdot[\gamma]$ is contractible in M. Similarly, $(m+1)^n\cdot[\gamma]$ is contractible in M, and we are done, because m^n and $(m+1)^n$ are coprime.

(ii) Since an affine nonsingular algebraic hypersurface in $\mathbb{C}^{n+1}$, $n\ge 2$, is simply connected, the previous argument shows that $\hat{M}\backslash\mathbb{C}P^n\subset\mathbb{C}^{n+1}$ is simply connected,

where $\hat{M}$ is a C-hypersurface in $\mathbb{C}P^{n+1}$ and $\mathbb{C}P^n$ is a coordinate hyperplane in $\mathbb{C}P^{n+1}$.

Let a C-hypersurface M of degree d in $\mathbb{C}P^n$ be defined by a subdivision $\mathcal{S}:\ T_d^n = \Delta_1 \cup ... \cup \Delta_N$ and collection of numbers $\mathcal{A} : T_d^n \cap \mathbb{Z}^n \to \mathbb{C}$. Embed T_d^n into T_d^{n+1} as the face $T_d^{n+1} \cap \{i_{n+1} = 0\}$, take the subdivision $\widetilde{\mathcal{S}}:\ T_d^{n+1} = \widetilde{\Delta}_1 \cup ... \cup \widetilde{\Delta}_N$ with $\widetilde{\Delta}_k$ being the cone over Δ_k with the vertex $(0, ..., 0, d)$, $k = 1, ..., N$, and define $\widetilde{\mathcal{A}} : T_d^{n+1} \cap \mathbb{Z}^{n+1} \to \mathbb{C}$ by $\widetilde{A}_{i_1...i_n 0} = A_{i_1...i_n}$, $\widetilde{A}_{0...0d} = -1$, $\widetilde{A}_{i_1...i_n i_{n+1}} = 0$ if $0 < i_{n+1} < d$. These data define non-degenerate polynomials $\widetilde{F}_k(z_1, ..., z_{n+1}) = F_k(z_1, ..., z_n) - z_{n+1}^d$ with Newton polytopes $\widetilde{\Delta}_k$, $k = 1, ..., N$, whose charts can be glued into a C-hypersurface $\widetilde{M}$ of degree d in $\mathbb{C}P^{n+1}$. The required isomorphism $\pi_1(\mathbb{C}P^n \backslash M) = \mathbb{Z}/d\mathbb{Z}$ is a corollary of the following statement.

Lemma 2.15 *There is a $\mathbb{Z}/d\mathbb{Z}$-covering $\widetilde{M} \backslash \{z_{n+1} = 0\} \to \mathbb{C}P^n \backslash M$.*

Proof. We construct a $\mathbb{Z}/d\mathbb{Z}$-covering $\Phi : \mathbb{C}Ch(\widetilde{\mathcal{S}}, \widetilde{\mathcal{A}}) \backslash \{w_{n+1} = 0\} \to \mathbb{C}T_d^n \backslash \mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ and then show that it commutes with the S^1-action on $\partial\mathbb{C}Ch(\widetilde{\mathcal{S}}, \widetilde{\mathcal{A}}) \backslash \{z_{n+1} = 0\}$ and $\partial\mathbb{C}T_d^n \backslash \mathbb{C}Ch(\mathcal{S}, \mathcal{A})$, thereby defining the required covering.

Let $(w_1, ..., w_n, w_{n+1}) \in \mathbb{C}Ch(\widetilde{\mathcal{S}}, \widetilde{\mathcal{A}})$, $w_{n+1} \neq 0$. Then $(w_1, ..., w_{+1}) \in \mathbb{C}I(\widetilde{\delta})$, where $\widetilde{\delta}$ is a cone over a face δ of Δ_k, $1 \le k \le N$, with the vertex at $(0, ..., 0, d)$. Hence $(w_1, ..., w_{n+1}) = \mathbb{C}\mu_{\widetilde{\delta}}(z_1, ..., z_{n+1})$ for some $(z_1, ..., z_{n+1}) \in (\mathbb{C}^*)^{n+1}$. So, we define

$$\Phi(w_1, ..., w_{n+1}) = \mathbb{C}\mu_\delta(z_1, ..., z_n)\ .$$

The map Φ is well defined. If $\delta = \Delta_1, \ldots, \Delta_N$, then $\mathbb{C}\mu_{\widetilde{\delta}}$ and $\mathbb{C}\mu_\delta$ are diffeomorphisms. If δ lies in an $(n-1)$-plane $\alpha_1 i_1 + ... + \alpha_n i_n = \beta$, $i_{n+1} = 0$, then $\widetilde{\delta}$ lies in an n plane $\alpha_1 i_1 + ... + \alpha_n i_n + \beta i_n/d = \beta$. This means that $(\mathbb{C}\mu_{\widetilde{\delta}})^{-1}(w_1, ..., w_{n+1})$ contains the family $(z_1 t^{\alpha_1}, ..., z_n t^{\alpha_n}, z_{n+1}^{\beta/d})$, $t \in \mathbb{R}_+^*$, but $\mathbb{C}\mu_\delta$ takes the family $(z_1 t^{\alpha_1}, ..., z_n t^{\alpha_n})$, $t \in \mathbb{R}_+^*$, to one point.

The map Φ is continuous. If, for some $k = 1, ..., N$ and a curve $\widetilde{\gamma}(t) \in (\mathbb{C}^*)^{n+1}$:

$$(\lambda_1 t^{k_1} + O(t^{k_1+1}), ..., \lambda_n t^{k_n} + O(t^{k_n+1}), \lambda_{n+1} t^{k_{n+1}} + O(t^{k_{n+1}+1})), \quad t > 0,$$

one has $\lim_{t\to 0} \mathbb{C}\mu_{\widetilde{\Delta}_k}(\widetilde{\gamma}(t)) = \mathbb{C}\mu_{\widetilde{\delta}}(\lambda_1, ..., \lambda_n, \lambda_{n+1})$, then $\lim_{t\to 0} \mathbb{C}\mu_{\Delta_k}(\gamma(t)) = \mathbb{C}\mu_\delta(\lambda_1, ..., \lambda_n)$, where

$$\gamma(t) = (\lambda_1 t^{k_1} + O(t^{k_1+1}), ..., \lambda_n t^{k_n} + O(t^{k_n+1})) \in (\mathbb{C}^*)^n, \quad t > 0\ .$$

The map Φ is surjective. Indeed, if $(w_1, ..., w_n) \in \mathbb{C}T_d^n \backslash \mathbb{C}Ch(\mathcal{S}, \mathcal{A})$, then $(w_1, ..., w_n) = \mathbb{C}\mu_\delta(z_1, ..., z_n)$ for a face δ of Δ_k ($1 \le k \le N$), where $(z_1, ..., z_n) \in (\mathbb{C}^*)^n$, $F_k^\delta(z_1, ..., z_n) \neq 0$. Then there exists $z_{n+1} \neq 0$ such that $F_k^\delta(z_1, ..., z_n) = z_{n+1}^d$; hence $\mathbb{C}\mu_{\widetilde{\delta}}(z_1, ..., z_n, z_{n+1})$ belongs to $\mathbb{C}Ch(F_k^{\widetilde{\delta}}) \backslash \{w_{n+1} = 0\}$ and $\Phi(\mathbb{C}\mu_{\widetilde{\delta}}(z_1, ..., z_n, z_{n+1})) = (w_1, ..., w_n)$.

The map Φ is a $\mathbb{Z}/d\mathbb{Z}$-covering. Indeed, for any point $(w_1, ..., w_n) \in \mathbb{C}T_d^n \backslash \mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ its preimage $\Phi^{-1}(w_1, ..., w_n)$ consists of d distinct points $(w_1', ..., w_n', w_{n+1}'\omega)$ with some fixed $w_1', ..., w_{n+1}'$ and any d-th root of unity ω.

At last, note that if $\Phi(w_1',...,w_n',w_{n+1}') = (w_1,...,w_n)$, then $\Phi(w_1'v,...,w_n'v,w_{n+1}'v) = (w_1v,...,w_nv)$ for arbitrary $v \in S^1$, which completes the proof. $\square$

Proposition 2.16 *Let $M \subset \mathbb{C}P^n$ be a C-hypersurface and $M' \subset \mathbb{C}P^n$ a nonsingular algebraic hypersurface, both of degree d. Then $H_*(M)$ and $H_*(M')$ are isomorphic as graded groups.*

Proof. Take m_0 as in Proposition 2.10. Choose $m > m_0$ in such a way that m is coprime with all the orders of elements in $\mathrm{Tors}H_*(M)$, and consider $M_m = \Pi_m^{-1}(M)$. Pick a homology class $\alpha \in H_i(M)$. Note that $m^n\alpha$ belongs to the image of $(\Pi_m)_* : H_i(M_m) \to H_i(M)$. For an odd i different from $n-1$, we have $H_i(M_m) = 0$ and, hence, $H_i(M) = 0$. In particular, $H_*(M)$ has no torsion. Any group $H_i(M)$ with even i ($0 \le i \le 2n-2$) contains a nontrivial element: the fundamental class of the intersection of M with coordinate hyperplanes taken in appropriate number. Thus, $H_i(M)$ is isomorphic to $H_i(M_m) \simeq \mathbb{Z}$ if i is even and different from $n-1$. The fact that the groups $H_{n-1}(M)$ and $H_{n-1}(M')$ are isomorphic follows now from the equality $\chi(M) = \chi(M')$ proven in Corollary 2.13. $\square$

Proposition 2.17 *Let n be a positive odd number, M a C-hypersurface of degree d in $\mathbb{C}P^n$ and M' a nonsingular algebraic hypersurface of degree d in $\mathbb{C}P^n$. Then the lattices $(H_{n-1}(M), B_M)$ and $(H_{n-1}(M'), B_{M'})$ (where $B_M : H_{n-1}(M) \times H_{n-1}(M) \to \mathbb{Z}$ and $B_{M'} : H_{n-1}(M') \times H_{n-1}(M') \to \mathbb{Z}$ are the intersection forms on M and M', respectively) are isomorphic.*

Proof. The lattices $H_{n-1}(M)$ and $H_{n-1}(M')$ are unimodular and have the same rank and signature (Corollary 2.13). It remains to show that these two lattices have the same parity, since for $d = 1$ we have $H_{n-1}(M) \simeq H_{n-1}(M') \simeq \mathbb{Z}$, and for $d \ge 2$ the lattice $H_{n-1}(M')$ is indefinite. Let us show, first, that the lattices $H_{n-1}(M)$ and $H_{n-1}(\widetilde{M})$ have the same parity, where $\widetilde{M} = \Pi_m^{-1}(M)$, m_0 is as in Proposition 2.10 and $m > m_0$ is odd.

(i) If α^2 is odd, $\alpha \in H_{n-1}(M)$, then $((\Pi_m)^!\alpha)^2 = m^{2n}\alpha^2$ is odd as well.

(ii) If β^2 is odd, $\beta \in H_{n-1}(\widetilde{M})$, then γ^2 is odd, where $\gamma = \sum_{g \in G} g_*\beta$, and $G = (\mathbb{Z}/m)^n$ is the deck transformation group of Π_m. Furthermore, $\gamma = (\Pi_m)^!\alpha$ for certain $\alpha \in H_{n-1}(M)$, and α^2 is odd.

Similarly, the lattices $H_{n-1}(M')$ and $H_{n-1}(\widetilde{M}')$, where $\widetilde{M}' = \Pi_m^{-1}(M')$, have the same parity. Since $\widetilde{M}$ and $\widetilde{M}'$ are isotopic in $\mathbb{C}P^n$, we obtain that the lattices $H_{n-1}(M)$ and $H_{n-1}(M')$ have the same parity. $\square$

Corollary 2.18 *A C-surface of degree d in $\mathbb{C}P^3$ is homeomorphic to a nonsingular algebraic surface of degree d in $\mathbb{C}P^3$.*

Proof. Note that a C-surface of degree d in $\mathbb{C}P^3$ is simply connected (see 2.13), smoothable (see Proposition 2.7) and has the same intersection form as a nonsingular algebraic surface of degree d in $\mathbb{C}P^3$ (see 2.17). It remains to apply Freedman's theorem (see, for example, [11]). $\Box$

2.4 Ramified double coverings

Proposition 2.19 *Let M be a C-hypersurface of even degree d in $\mathbb{C}P^n$. Then there exists a closed simply connected $2n$-dimensional PL-manifold Y and a map $\Pi : Y \to \mathbb{C}P^n$ which is a double covering ramified along M.*

Proof. The existence of a ramified double covering $\Pi : Y \to \mathbb{C}P^n$ with Y being a PL-manifold follows generically from the fact that $[M] \in H_{2n-2}(\mathbb{C}P^n)$ is an even class. However, we prefer to give an explicit construction of Y.

Namely, we repeat the construction of step (ii) in the proof of Corollary 2.14, taking the simplex $T^{n+1}_{d,2} \subset \mathbb{R}^{n+1}$ with vertices $(0, ..., 0)$, $(d, 0, ..., 0)$, ..., $(0, ..., 0, d, 0)$, $(0, ..., 0, 2)$ instead of T^{n+1}_d. So, T^n_d embeds into $T^{n+1}_{d,2}$ as the face $T^{n+1}_{d,2} \cap \{i_{n+1} = 0\}$, the subdivision $T^{n+1}_{d,2} = \widetilde{\Delta}_1 \cup ... \cup \widetilde{\Delta}_N$ is defined as the cone over the subdivision $T^n_d = \Delta_1 \cup ... \cup \Delta_N$ with the vertex at $(0, ..., 0, 2)$, and $\widetilde{\mathcal{A}} : T^{n+1}_{d,2} \cap \mathbb{Z}^{n+1} \to \mathbb{C}$ is defined by $\mathcal{A}_{i_1...i_n0} = A_{i_1...i_n}$, $\widetilde{\mathcal{A}}_{i_1...i_n1} = 0$, $\widetilde{\mathcal{A}}_{0...02} = -1$. These data define a PL-manifold $\mathbb{C}Ch(\widetilde{\mathcal{S}}, \widetilde{\mathcal{A}})$ with boundary $\partial\mathbb{C}Ch(\widetilde{\mathcal{S}}, \widetilde{\mathcal{A}}) = \mathbb{C}Ch(\widetilde{\mathcal{S}}, \widetilde{\mathcal{A}}) \cap \partial\mathbb{C}T^{n+1}_{d,2}$, which is the union of the charts of the polynomials $\widetilde{F}_k(z_1, ..., z_n, z_{n+1}) = F_k(z_1, ..., z_n) - z^2_{n+1}$, $k = 1, ..., N$, and, as in the proof of Lemma 2.15, there exists a double covering $\Phi : \mathbb{C}Ch(\widetilde{\mathcal{S}}, \widetilde{\mathcal{A}}) \to \mathbb{C}T^n_d$ ramified along $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$.

Now note that S^1 acts on $\partial\mathbb{C}T^{n+1}_{d,2}$ by

$$v \in S^1,\ (w_1, ..., w_n, w_{n+1}) \in \partial\mathbb{C}T^{n+1}_{d,2} \ \mapsto\ (w_1 v, ..., w_n v, w_{n+1} v^{d/2}) \in \partial\mathbb{C}T^{n+1}_{d,2}\ .$$

The manifold $\partial\mathbb{C}Ch(\widetilde{\mathcal{S}}, \widetilde{\mathcal{A}})$ is invariant with respect to this action, because it is defined by quasi-homogeneous polynomials $f(z_1, ..., z_{n+1})$ satisfying

$$f(z_1\tau, ..., z_n\tau, z_{n+1}\tau^{d/2}) = \tau^d f(z_1, ..., z_{n+1}),$$

which is compatible with the S^1-action. By the same reason

$$\Phi(w_1 v, ..., w_n v, w_{n+1} v^{d/2}) = (w'_1 v, ..., w'_n v), \quad v \in S^1\ ,$$

as far as $\Phi(w_1, ..., w_n, w_{n+1}) = (w'_1, ..., w'_m)$, $(w_1, ..., w_{n+1}) \in \partial\mathbb{C}Ch(\widetilde{\mathcal{S}}, \widetilde{\mathcal{A}})$. Hence Φ reduces to a double covering $\Pi : Y \to \mathbb{C}P^n$, where $Y = \mathbb{C}Ch(\widetilde{\mathcal{S}}, \widetilde{\mathcal{A}})/S^1$, ramified along M. It remains to show that Y is a closed PL-manifold. Indeed, for any edge of $T^{n+1}_{d,2}$ there exists a combination of an automorphism of $\mathbb{Z}^{n+1}$, leaving the $(n+1)$-st axis fixed, and shifts which puts this edge on a coordinate axis and the adjacent faces of $T^n_{d,s}$ on the corresponding coordinate planes. One can easily verify that the orbits of the S^1-action on that edge and adjacent faces contract into

points, whose neighborhoods in Y are homeomorphic to $\mathbb{R}^{2n}$ as shown in the proof of Proposition 1.17.

At last, we show that $\pi_1(Y)=0$. Any loop in $\mathbb{C}P^n$ through a point $\overline{w}\in M$ lifts to a loop on Y through $\overline{w}$, because $\overline{w}$ is covered in Y by itself only. Since any loop in $\mathbb{C}P^n$ is contractible, so does its lifting in Y. $\Box$

Proposition 2.20 *Let n and d be positive even numbers, M be a C-hypersurface of degree d in $\mathbb{C}P^n$, and Y be a double covering of $\mathbb{C}P^n$ ramified along M. Then*

$$\chi(Y)=\chi^n_{d,2},\quad \operatorname{sign}(Y)=\operatorname{sign}^n_{d,2},\quad H_*(Y)\simeq H_*(Y'),$$

and the lattices $H_n(Y)$ and $H_n(Y')$, equipped with the intersection forms, are isomorphic, where Y' is the double covering of $\mathbb{C}P^n$ ramified along a nonsingular algebraic hypersurface of degree d, and $\chi^n_{d,2}$ and $\operatorname{sign}^n_{d,2}$ are, respectively, the Euler characteristic and the signature of Y'.

Proof. The equalities $\chi(Y)=\chi(Y')$ and $\operatorname{sign}(Y)=\operatorname{sign}(Y')$ follow immediately from the additivity of the Euler characteristic, the Hirzebruch formula for the signature of ramified coverings and Corollary 2.13. The isomorphism $H_*(Y)\simeq H_*(Y')$, by [9], reduces to $H_{2i-1}(Y)=0$, $i=1,...,n$, $H_{2i}(Y)=\mathbb{Z}$, $i=0,...,n$, $i\ne n/2$, $H_n(Y)\simeq\mathbb{Z}^r$, $r=\mathrm{rk}H_n(Y')$. It can be proven as Proposition 2.16, using a ramified covering of Y similar to that constructed in Proposition 2.10. The same ramified covering applied as in the proof of Proposition 2.17, gives a lattice isomorphism $H_n(Y)\simeq H_n(Y')$. $\Box$

Proposition 2.21 *Let n and d be positive even numbers and M be a real C-hypersurface of degree d in $\mathbb{C}P^n$. Then $\mathbb{R}P^n=\mathbb{R}M_+\cup\mathbb{R}M_-$, where $\mathbb{R}M_+$ and $\mathbb{R}M_-$ are compact manifolds with boundary such that*

$$\partial\mathbb{R}M_+=\partial\mathbb{R}M_-=\mathbb{R}M_+\cap\mathbb{R}M_-=\mathbb{R}M\ .$$

The ramified double covering $\Pi:Y\to\mathbb{C}P^n$ admits an action of the group $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}=\{\mathrm{Id},\tau,\mathrm{Conj}_+,\mathrm{Conj}_-\}$, where τ is the deck transformation of Π and $\Pi\circ\mathrm{Conj}_+=\Pi\circ\mathrm{Conj}_-=\mathrm{Conj}\circ\Pi$. In addition,

$$\mathrm{Fix}(\tau)=M,\quad \mathbb{R}Y_\pm\overset{\text{def}}{=}\mathrm{Fix}(\mathrm{Conj}_\pm)=\Pi^{-1}(\mathbb{R}M_\pm)\ .$$

Proof. In the framework of the construction in the proof of Proposition 2.19, we define $\mathbb{R}Ch(\mathcal{S},\mathcal{A})_+$ (resp., $\mathbb{R}Ch(\mathcal{S},\mathcal{A})_-$) as the union of the closures of $\mathbb{C}\mu_{\Delta_k}(\{F_k\ge 0\}\cap(\mathbb{R}^*)^n)$ (resp., $\mathbb{C}\mu_{\Delta_k}(\{F_k\le 0\}\cap(\mathbb{R}^*)^n)$), $k=1,...,N$. The action of S^1 on $\partial\mathbb{C}T^n_d$ reduces to the action of $S^0=\{\pm1\}$ on $\partial\mathbb{R}T^n_d$ which preserves $\mathbb{R}Ch(\mathcal{S},\mathcal{A})_+$ and $\mathbb{R}Ch(\mathcal{S},\mathcal{A})_-$, thus defining $\mathbb{R}M_\pm=\mathbb{R}Ch(\mathcal{S},\mathcal{A})_\pm/S^0$. The involutions

$$(w_1,...,w_n,w_{n+1})\mapsto(w_1,...,w_n,-w_{n+1})\ ,$$
$$(w_1,...,w_n,w_{n+1})\mapsto\mathrm{Conj}(w_1,...,w_n,w_{n+1})\ ,$$
$$(w_1,...,w_n,w_{n+1})\mapsto\mathrm{Conj}(w_1,...,w_n,-w_{n+1})$$

on the double covering $\mathbb{C}Ch(\widetilde{\mathcal{S}}, \widetilde{\mathcal{A}})$ of $\mathbb{C}T^n_d$ ramified along $\mathbb{C}Ch(\mathcal{S}, \mathcal{A})$ induce the involutions τ, Conj_+ and Conj_-, respectively, on Y. $\square$

Now we have to make a digression on the topology of smooth (not necessarily algebraic) hypersurfaces in $\mathbb{R}P^n$. Following V. Kharlamov [21] and O. Viro [35], we define a *rank* of a connected smooth hypersurface in $\mathbb{R}P^n$ to be the maximal integer r such that the homomorphism induced in r-dimensional homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients by the inclusion of the hypersurface into the projective space is nontrivial. It is easy to see that the intersection of a hypersurface of rank r in $\mathbb{R}P^n$ with a transversal projective subspace P of dimension $k \geq n - r$ is a hypersurface of rank $r - n + k$ in P. Similarly, we define a *rank* of a connected component of the complement of a hypersurface in $\mathbb{R}P^n$. Clearly, the rank of a connected component C of the complement of a two-sided hypersurface (*i.e.*, a hypersurface dividing its tubular neighborhood) is greater or equal to the rank of each component of the boundary ∂C.

A component C of the complement of a two-sided hypersurface is called *principal* if the rank of C is greater then the rank of each component of ∂C.

Proposition 2.22 *Let S be a two-sided hypersurface in $\mathbb{R}P^n$. Then there exists at most one principal component of $\mathbb{R}P^n \setminus S$.*

Proof. Suppose that $\mathbb{R}P^n \setminus S$ has two principal components C_1 and C_2. Let r_1 (resp., r_2) be the maximal rank of a connected component of ∂C_1 (resp., ∂C_2). Assume that $r_1 \leq r_2$. Consider a projective subspace P of $\mathbb{R}P^n$ transversal to S and of dimension $n - r_1$. Each connected component of $P \cap \partial C_1$ is of rank 0 in P and divides P into two parts. One of these parts has rank 0 in P, and is called the *interior* of the component. Denote by C the unique component of $P \setminus \partial C_1$ which is not contained in the interior of any component of $P \cap \partial C_1$. Note that $P \cap C_1$ should have a connected component of positive rank. Therefore $C \subset P \cap C_1$. It implies that $P \cap C_2$ is contained in the interior of some component of $P \cap \partial C_1$, and thus should have rank 0. $\square$

Clearly, the real parts of a real algebraic hypersurface of even degree d and of a real C-hypersurface of even degree d are two-sided in $\mathbb{R}P^n$. Switching if necessary $\mathbb{R}M_+$ and $\mathbb{R}M_-$, we suppose from now on that $\mathbb{R}M_-$ contains the principal component of $\mathbb{R}P^n \setminus \mathbb{R}M$, if this principal component does exist.

Denote by $\mathrm{def}(\mathbb{R}M)$ the dimension of the intersection of the kernel of $\frown \omega$: $H_*(\mathbb{R}M_-, \mathbb{R}M; \mathbb{Z}/2\mathbb{Z}) \to H_*(\mathbb{R}M_-, \mathbb{R}M; \mathbb{Z}/2\mathbb{Z})$, defined by $\sigma \mapsto \sigma \frown \omega$, where ω is $(d/2)$-times the generator of $H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$, with the kernel of the boundary homomorphism $H_*(\mathbb{R}M_-, \mathbb{R}M; \mathbb{Z}/2\mathbb{Z}) \to H_*(\mathbb{R}M; \mathbb{Z}/2\mathbb{Z})$. We call this dimension $\mathrm{def}(\mathbb{R}M)$ the *defect* of $\mathbb{R}M$. For any manifold X denote by $b_*(X)$ the total Betti number $\dim_{\mathbb{Z}/2\mathbb{Z}} H_*(X; \mathbb{Z}/2\mathbb{Z})$ of X.

Proposition 2.23 *Let n be a positive even number, and M a real C-hypersurface of even degree d in $\mathbb{C}P^n$. Then $b_*(\mathbb{R}Y_+) = b_*(\mathbb{R}M)$ and $b_*(\mathbb{R}Y_-) = b_*(\mathbb{R}M) + 2 \cdot \mathrm{def}(\mathbb{R}M)$.*

Proof. The statement can be easily derived from the Smith exact sequence (see, for example, [4, 38]) applied to the deck transformation of $\mathbb{R}Y_\pm$ (as it is done in [26], [20] and [7] in the case of real algebraic hypersurfaces). $\Box$

Remark 2.24 *Note that if n and d are even, then $\mathbb{R}P^n \backslash \mathbb{R}M$ should have a principal component. Indeed, if $\mathbb{R}P^n \setminus \mathbb{R}M$ does not have a principal component, one has $b_*(\mathbb{R}Y_+) = b_*(\mathbb{R}M)$ and $b_*(\mathbb{R}Y_-) = b_*(\mathbb{R}M)$, and thus, $\chi(\mathbb{R}Y_+) \equiv \chi(\mathbb{R}Y_-) \mod 4$. The last congruence is impossible, because $\chi(\mathbb{R}Y_\pm) = 2\chi(\mathbb{R}M_\pm)$ and $\chi(\mathbb{R}M_+) + \chi(\mathbb{R}M_-) = \chi(\mathbb{R}P^n) = 1$.*

3 Topology of real C-hypersurfaces

3.1 Generalized Harnack inequalities

Let M' be a nonsingular algebraic hypersurface of degree d in $\mathbb{C}P^n$. Denote by b^n_d the total Betti number $b_*(M') = \dim_{\mathbb{Z}/2\mathbb{Z}} H_*(M'; \mathbb{Z}/2\mathbb{Z})$ of M'. One has (see, for example, [6])

$$b^n_d = \frac{(d-1)^{n+1} - (-1)^{n+1}}{d} + d + (-1)^{n+1}.$$

Theorem 3.1 *For a real C-hypersurface M of degree d in $\mathbb{C}P^n$, one has*

$$b_*(\mathbb{R}M) = b^n_d - 2a(\mathbb{R}M), \tag{21}$$

where $a(\mathbb{R}M)$ is a nonnegative integer. Furthermore, if n and d are both even, then

$$b_*(\mathbb{R}Y_-) = b^n_{d,2} - 2a(\mathbb{R}M) + 2(\mathrm{def}(\mathbb{R}M) - 1) \leq b^n_{d,2}, \tag{22}$$

$$b_*(\mathbb{R}Y_+) = b^n_{d,2} - 2a(\mathbb{R}M) - 2, \tag{23}$$

where Y is the double covering of $\mathbb{C}P^n$ ramified along M, $\mathbb{R}Y_\pm$ are the fixed point sets of two liftings $\mathrm{Conj}_\pm$ to Y of the complex conjugation Conj in $\mathbb{C}P^n$, and $b^n_{d,2}$ is the total Betti number of the double covering of $\mathbb{C}P^n$ ramified along a nonsingular hypersurface of degree d.

Proof. The statement follows from the Smith-Floyd inequality

$$\dim_{\mathbb{Z}/2\mathbb{Z}} H_*(\mathrm{Fix}(\tau); \mathbb{Z}/2\mathbb{Z}) \leq \dim_{\mathbb{Z}/2\mathbb{Z}} H_*(X; \mathbb{Z}/2\mathbb{Z}) \tag{24}$$

(where X is a compact CW-complex and $\tau : X \to X$ is an involution) and the congruence

$$\dim_{\mathbb{Z}/2\mathbb{Z}} H_*(X; \mathbb{Z}/2\mathbb{Z}) \equiv \dim_{\mathbb{Z}/2\mathbb{Z}} H_*(\mathrm{Fix}(\tau); \mathbb{Z}/2\mathbb{Z}) \mod 2; \tag{25}$$

see for details [26, 38]. We apply the Smith-Floyd inequality and the above congruence to the involutions Conj on M and $\mathrm{Conj}_\pm$ on Y. In order to get (21) we

use Proposition 2.16. To prove (22) and (23) we notice that according to Proposition 2.23 one has $b_*(\mathbb{R}Y_+) = b_*(\mathbb{R}M)$ and $b_*(\mathbb{R}Y_-) = b_*(\mathbb{R}M) + 2\mathrm{def}(\mathbb{R}M)$. In addition, since n is even,

$$b_*(Y) = \chi(Y) = 2\chi(\mathbb{C}P^n) - \chi(M) = 2(n+1) - (2n - b_*(M)) = 2 + b_*(M) \ ,$$

and it remains to apply Proposition 2.20. $\square$

3.2 Congruences

Theorem 3.2 *Let M be a real C-hypersurface of degree d in $\mathbb{C}P^n$. If n is odd then*

$$\chi(\mathbb{R}M) \equiv \mathrm{sign}_d^n + \begin{cases} 0, & \text{if } a(\mathbb{R}M) = 0, \\ \pm 2, & \text{if } a(\mathbb{R}M) = 1 \end{cases} \mod 16 \ . \tag{26}$$

If n and d are even, then

$$\chi(\mathbb{R}M_-) \equiv \frac{\mathrm{sign}_{d,2}^n}{2} + \begin{cases} 0, & \text{if } a(\mathbb{R}M) = \mathrm{def}(\mathbb{R}M) - 1, \\ \pm 1, & \text{if } a(\mathbb{R}M) = \mathrm{def}(\mathbb{R}M) \end{cases} \mod 8 \ , \tag{27}$$

$$\chi(\mathbb{R}M_+) \equiv \frac{\mathrm{sign}_{d,2}^n}{2} \pm 1 \mod 8, \ \text{if } a(\mathbb{R}M) = 0. \tag{28}$$

Remark 3.3 *Note that in the case of even n and d the condition $a(\mathbb{R}M) = \mathrm{def}(\mathbb{R}M) - 1$ is automatically fulfilled if $a(\mathbb{R}M) = 0$, and the condition $a(\mathbb{R}M) = \mathrm{def}(\mathbb{R}M)$ is automatically fulfilled if $a(\mathbb{R}M) = 1$.*

Proof of Theorem 3.2. The statement can be proven as the Rokhlin and Gudkov-Krahnov-Kharlamov congruences in the algebraic case [14, 20, 26, 38].

(i) Let $n = 2k+1$ and $a(\mathbb{R}M) = 0$. As in [26] (see also [14, 38]), the latter implies the splitting of $H_{n-1}(M)$ into the orthogonal sum $H_+ \oplus H_-$ of unimodular eigenlattices corresponding to the eigenvalues ± 1 of $\mathrm{Conj}_* : H_{n-1}(M) \to H_{n-1}(M)$.

Lemma 3.4 *The signature of involution $\mathrm{sign}(\mathrm{Conj}_*) = \mathrm{sign}(H_+) - \mathrm{sign}(H_-)$ is equal to $(-1)^k\chi(\mathbb{R}M)$.*

Proof. By the Atiyah-Singer formula (see [3, 26, 38]) $\mathrm{sign}(\mathrm{Conj}_*)$ is equal to $\mathbb{R}M \circ \mathbb{R}M$, the self-intersection of $\mathbb{R}M$ in M. Let us show that $\mathbb{R}M \circ \mathbb{R}M = (-1)^k\chi(\mathbb{R}M)$. First, we smooth M as in Proposition 2.7. Then we take a tangent vector field V on $\mathbb{R}M_{sm}$ having only finitely many singular points which are all non-degenerate and lie outside $\mathbb{R}\delta$ for any proper face δ of the polytopes $\Delta_1, ..., \Delta_N$ in the subdivision of T_d^n. Extend the vector field $J(V)$ (where J is the almost complex structure on M_{sm} defined in Proposition 2.8) to a neighborhood of $\mathbb{R}M_{sm}$ in M_{sm} and slightly move $\mathbb{R}M_{sm}$ along geodesics in M_{sm} tangent to the field obtained. The result has transversal intersection points with $\mathbb{R}M_{sm}$ at the singular points of V.

The intersection indices are equal to the multiplied by $(-1)^k$ indices of the singular points of V (see [26, 38]). $\Box$

Denote by H_e the lattice H_+ (resp., H_-) if k is odd (resp., k is even). From the existence of an almost complex structure on a smoothing of M (Proposition 2.8), it follows that H_e is even (see, for example, [38]). Let σ_e be the signature of H_e. Since H_e is unimodular and even, the signature σ_e is divisible by 8. To finish the proof in the case $a(\mathbb{R}M) = 0$, it remains to note that according to Lemma 3.4 we have $\mathrm{sign}(M) - \chi(\mathbb{R}M) = 2\sigma_e$.

The prove (27) in the case $a(\mathbb{R}M) = \mathrm{def}(\mathbb{R}M) - 1$, we apply the same arguments to (Y, Conj_-) and use the relations $\chi(\mathbb{R}Y_-) = 2\chi(\mathbb{R}M_-)$ and $b_*(\mathbb{R}Y_-) = b_*(\mathbb{R}M) + 2\mathrm{def}(\mathbb{R}M)$.

(ii) Let now $n = 2k+1$ and $a(\mathbb{R}M) = 1$. In this case (see, for example, [20, 38]), the discriminants of H_+ and H_- are ± 2 (from the Smith theory we get the inequality $\mid \mathrm{discr}(H_\pm) \mid \leq 2$ and then use the congruence $\chi(M) \equiv (-1)^k \mathrm{sign}(M) \mod 4$ to show that $\mid \mathrm{discr}(H_\pm) \mid \neq 1$). Using Lemma 3.4 and the fact that the signature of an even lattice with discriminant ± 2 is congruent to $\pm 1 \mod 8$ we immediately obtain the statement required.

To prove (27) in the case $a(\mathbb{R}M) = \mathrm{def}(\mathbb{R}M)$ and (28) we again apply the previous arguments to (Y, Conj_-) and (Y, Conj_+). $\Box$

3.3 Comessatti inequality for real C-surfaces

Theorem 3.5 *Let M be a real C-hypersurface of degree d in $\mathbb{C}P^n$ and M' be a nonsingular algebraic hypersurface of degree d in $\mathbb{C}P^n$. Then*

$$2 - h^{1,1}(M') \leq \chi(\mathbb{R}M) \leq h^{1,1}(M').$$

Proof. The arguments are completely similar to the proof of the Comessatti inequality in the case of real algebraic surfaces.

Let H_+ and H_- be again eigenlattices of $H_2(M)$ corresponding to the eigenvalues ± 1 of $\mathrm{Conj}_* : H_2(M) \to H_2(M)$. Denote by $a^+_\pm$ (resp., $a^-_\pm$) the number of positive (resp., negative) squares in the diagonal form over $\mathbb{Q}$ of the restriction of $B : H_2(M) \times H_2(M) \to \mathbb{Z}$ to $H_\pm$. We have

$$\begin{gathered} a^+_+ + a^-_+ + a^+_- + a^-_- = \dim H_2(M), \\ a^+_+ - a^-_+ + a^+_- - a^-_- = \mathrm{sign}(M), \\ a^+_+ + a^-_+ - a^+_- - a^-_- = \chi(\mathbb{R}M) - 2, \\ a^+_+ - a^-_+ - a^+_- + a^-_- = -\chi(\mathbb{R}M). \end{gathered}$$

The third and fourth equalities follow from the Lefschetz fixed point theorem and Atiyah-Singer theorem, respectively. We obtain that

$$4a^-_+ = \dim H_2(M) - \mathrm{sign}(M) + 2\chi(\mathbb{R}M) - 2 \geq 0,$$

$$4a^-_- = \dim H_2(M) - \operatorname{sign}(M) - 2\chi(\mathbb{R}M) + 2 \geq 0.$$

These inequalities together with the Hodge index relations give the required statement. □

3.4 Topology of real C-curves

Let M be an oriented smooth connected closed surface in $\mathbb{C}P^2$. Then M is called a *flexible curve of degree* d (see [33]) if

- it realizes $d[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$,
- the genus of M is equal to $(d-1)(d-2)/2$,
- M is invariant under the complex conjugation,
- the field of tangent planes to M on $M \cap \mathbb{R}P^2$ can be equivariantly deformed to the field of lines in $\mathbb{C}P^2$ tangent to $M \cap \mathbb{R}P^2$.

According to Propositions 2.6, 2.8 and Corollary 2.13 (a smoothing of) a real C-curve of degree d in $\mathbb{C}P^2$ is a flexible curve of degree d. Thus, all the restrictions on the topology of flexible curves are applicable to real C-curves. We formulate here in the framework of real C-curves the principal known restrictions on flexible curves. An extensive list of restrictions to the topology of flexible curves can be found in [33].

Let us start from definitions. The standard definitions applicable to real algebraic curves can be naturally extended to real C-curves. A real C-curve A of degree d in $\mathbb{C}P^2$ is called an *M-curve* or *maximal* if the real part $\mathbb{R}A$ of A has $(d-1)(d-2)/2+1$ connected components. A real C-curve A of degree d in $\mathbb{C}P^2$ is called an *$(M-i)$-curve* if $\mathbb{R}A$ has $(d-1)(d-2)/2+1-i$ connected components. A connected component of the real part of a real C-curve of degree d in $\mathbb{C}P^2$ is called an *oval* if it divides $\mathbb{R}P^2$ into two parts. The part homeomorphic to a disk is called the *interior* of the oval. All the connected components of the real part of a real C-curve of an even degree in $\mathbb{C}P^2$ are ovals. Exactly one connected component of the real part of a real C-curve of an odd degree in $\mathbb{C}P^2$ is not an oval. This component is called *nontrivial*. An oval is *even* (resp., *odd*) if it lies inside of an even (resp., odd) number of other ovals of the curve. The numbers of even and odd ovals of a curve are denoted by p and n, respectively. The Euler characteristic of a connected component of the complement in $\mathbb{R}P^2$ of the real part of a real C-curve is called the *characteristic* of an oval bounding the component from outside. A component of the complement in $\mathbb{R}P^2$ of the real part of a real C-curve is said to be *even* if each of its inner bounding ovals contains inside an odd number of ovals.

Theorem 3.6 • Harnack inequality. *The number of connected components of the real part of a real C-curve of degree d in $\mathbb{C}P^2$ is at most $(d-1)(d-2)/2+1$.*

- Gudkov-Rokhlin congruence. *For a maximal real C-curve of degree* $2k$ *in* $\mathbb{C}P^2$, *one has*
$$p - n \equiv k^2 \mod 8.$$

- Gudkov-Krahnov-Kharlamov congruence. *Let* A *be a real C-curve of degree* $2k$ *in* $\mathbb{C}P^2$. *If* A *is an* $(M-1)$*-curve, then*
$$p - n \equiv k^2 \pm 1 \mod 8.$$

- Strengthened Petrovsky inequalities. *For a real C-curve* A *of degree* $2k$ *in* $\mathbb{C}P^2$, *one has*
$$p - n^- \leq \frac{3k(k-1)}{2} + 1, \quad n - p^- \leq 3k(k-1),$$
where p^- *(resp.,* n^-*) is the number of even (resp., odd) ovals of* $\mathbb{R}A$ *with negative characteristic.*

- Strengthened Arnold inequalities. *For a real C-curve* A *of degree* $2k$ *in* $\mathbb{C}P^2$, *one has*
$$p^- + p^0 \leq \frac{k^2 - 3k + 3 + (-1)^k}{2}, \quad n^- + n^0 \leq \frac{k^2 - 3k + 2}{2},$$
where p^0 *(resp.,* n^0*) is the number of even (resp., odd) ovals of* $\mathbb{R}A$ *with characteristic* 0.

- Extremal properties of strengthened Arnold inequalities. *For a real C-curve of degree* $2k$ *in* $\mathbb{C}P^2$, *one has*
$p^- = p^+ = 0$, *if* k *is even and* $p^- + p^0 = (k^2 - 3k + 4)/2$,
$n^- = n^+ = 0$, *if* k *is odd and* $n^- + n^0 = (k^2 - 3k + 2)/2$.

A real C-curve A in $\mathbb{C}P^2$ is said to be of *type I* if its real part $\mathbb{R}A$ divides A into two parts; otherwise, the curve is of *type II*. For a curve A of type I, the orientations of two halves of $A \setminus \mathbb{R}A$ induce on $\mathbb{R}A$ two opposite orientations which are called *complex orientations*. Note that a real C-curve A is of type I if and only if all the algebraic curves used in the construction of A are of type I and complex orientations on the real parts of these curves can be chosen in such a way that they induce an orientation of $\mathbb{R}A$.

A pair of ovals of the real part of a real C-curve in $\mathbb{C}P^2$ is *injective* if one of them is inside of the other one. A collection of ovals is called a *nest* if any two of them form an injective pair. An injective pair of ovals of a real C-curve is *positive* (resp., *negative*) if the complex orientations of the ovals are induced (resp., are not induced) from some orientation of the annulus bounded by the ovals. Take an oval of a real C-curve of type I and of an odd degree in $\mathbb{C}P^2$, and consider the Möbius band which is the complement in $\mathbb{R}P^2$ of the interior of the oval. The oval is called *positive* (resp., *negative*) if the integer homology class realized in the Möbius band by the oval equipped with a complex orientation differs in sign (resp., coincides) with the class of the doubled nontrivial component equipped with the complex orientation.

Theorem 3.7 • Klein congruence. *Let A be a real C-curve of type I in $\mathbb{C}P^2$. If A is an $(M-i)$-curve, then $i \equiv 0 \mod 2$.*

- Arnold congruence. *For a real C-curve of type I and of degree $2k$, one has*

$$p - n \equiv k^2 \mod 4 .$$

- Rokhlin-Mishachev formulae. *For a real C-curve A of type I and of degree $2k$ in $\mathbb{C}P^2$, one has*

$$2(\Pi^+ - \Pi^-) = l - k^2,$$

where l is the number of ovals of $\mathbb{R}A$, and Π^+ and Π^- are the numbers of positive and negative injective pairs, respectively. For a real C-curve A of type I and of degree $2k+1$ in $\mathbb{R}P^2$, one has

$$2(\Pi^+ - \Pi^-) + \Lambda^+ - \Lambda^- = l - k(k+1),$$

where Λ^+ and Λ^- are the numbers of positive and negative ovals, respectively.

- Kharlamov- Marin congruence. *Let A be a real C-curve of degree $2k$ in $\mathbb{C}P^2$. If A is an $(M-2)$-curve and $p - n \equiv k^2 + 4 \mod 8$, then A is of type I.*

- Rokhlin inequalities. *Let A be a real C-curve of type I and of degree $2k$ in $\mathbb{C}P^2$. If k is even, then $4\nu + p - n \leq 2k^2 - 6k + 8$, where ν is the number of odd nonempty exterior bounding ovals of even components of $\mathbb{R}P^2 \setminus \mathbb{R}A$.*

  If k is odd, then $4\pi + n - p \leq 2k^2 - 6k + 7$, where π is the number of odd nonempty exterior bounding ovals of even components of $\mathbb{R}P^2 \setminus \mathbb{R}A$.

- Extremal properties of strengthened Arnold inequalities. *Let A be a real C-curve of degree $2k$ in $\mathbb{C}P^2$.*

  If k is even and $p^- + p^0 = (k^2 - 3k + 4)/2$, then A is of type I.

  If k is odd and $n^- + n^0 = (k^2 - 3k + 2)/2$, then A is of type I.

Harnack inequality in the case of real C-curves constructed using a primitive (*i.e.*, such that all its triangles are of area $1/2$) triangulation was, first, proved in [18] and then in a different way in [15]. Rokhlin-Mishachev formulae in the case of real C-curves constructed using a primitive triangulation was proved in [24].

It is interesting that one restriction on real C-curves is not proved in the case of flexible curves. This restriction was proved in [8]:

consider a real C-curve A of degree d in $\mathbb{C}P^2$; if A is constructed out of threenomials, then the sum of the depths of any two nests of $\mathbb{R}A$ is at most $d/2$.

In the case of real algebraic curves this statement is known as Hilbert's theorem and is an immediate corollary of the Bézout theorem.

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